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# On solvable models of type-IIB superstring in NS-NS and R-R plane wave backgrounds

**Jorge G. Russo**

*Departamento de Física, Universidad de Buenos Aires*
*Ciudad Universitaria and Conicet, Pab. I, 1428 Buenos Aires, Argentina*
*E-mail:* russo@df.uba.ar

**Arkady A. Tseytlin**

*Theoretical Physics Group, Blackett laboratory, Imperial College*
*London SW7 2BZ, U.K., and*
*Smith Laboratory, Ohio State University*
*Columbus OH 43210-1106, USA*
*E-mail:* tseytlin.1@osu.edu

Abstract: We consider type-IIB string in the two plane-wave backgrounds which may be interpreted as special limits of the $AdS_3 \times S^3$ supported by the NS-NS or R-R 3-form backgrounds. The NS-NS plane-wave string model is equivalent to a direct generalization of the Nappi-Witten model, with its spectrum being similar to that of strings in constant magnetic field. The R-R model can be solved in the light-cone gauge, where the Green-Schwarz world-sheet theory reduces to a system of free fields: 4 massive and 4 massless copies of bosons and fermions. We describe the string spectra of the two models and study the associated asymptotic density of states. We also discuss a more general class of exactly solvable plane-wave models with reduced or completely broken supersymmetry which are obtained by adding twists in two spatial 2-planes.

Keywords: Superstring Vacua, Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

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## 1. Introduction

Plane-wave backgrounds are exact solutions of string theory (see, e.g., [1]–[6] for a review). Certain NS-NS plane-wave backgrounds with non-zero 3-form field correspond to (gauged) WZW models [7]–[12]. More generally, a class of gravitational plane-wave backgrounds lead to exactly solvable (in terms of free oscillators in light-cone gauge) class of non-compact curved-space (super)string models, where one can explicitly find the string spectrum and compute simplest correlation functions in much the same way as in flat space. As was recently realised [13, 14, 15], this solvability property applies to string models corresponding not only to the NS-NS but also to certain R-R [16, 17] plane-wave backgrounds.

The simplest gravitational plane-wave metric

$$ds^2 = dudv - K(u,x)du^2 + dx_i dx_i$$

solves the vacuum Einstein (and, in fact, full string-theory) equations provided $\partial_i \partial^i K = 0$ ($i = 1, \ldots, D-2$). Particular solutions are $K_1 = w_{ij}(u) x_i x_j$ with $w_{ii} = 0$ or $K_2 = \frac{Q(u)}{|x|^{D-4}}$ (for $x \neq 0$). Fixing the light-cone gauge, $u = 2\alpha' p^u \tau$, we thus get the following effective lagrangian for the transverse coordinates:

$$L = \partial_+ x_i \partial_- x_i - V(\tau, x) , \qquad V = (\alpha' p^u)^2 K(2\alpha' p^u \tau, x) .$$

This gives a quadratic, and thus exactly solvable, string theory in the case of $K_1$ (but not in the case of $K_2$). For example, one may consider $K = w(u)(x_1^2 - x_2^2)$, solve explicitly the linear classical string equations, and then perform the canonical quantization (cf. [1, 4]).¡

The trace-free condition on the matrix $w_{ij}$ following from the Ricci-flatness of the metric implies that the corresponding quadratic potential $V$ is not positive definite in at least one of the directions.[1] It is possible to obtain models with non-negative $V$ by adding extra fields with constant null field strengths which produce non-vanishing ($uu$)-component of the stress tensor and thus change the condition $w_{ii} = 0$ into $w_{ii} = F^2 > 0$. As a result, the (non-trivial part of) corresponding string spectrum becomes discrete. For example, the Nappi-Witten (NW) [7] background which has a non-zero NS-NS 3-form field $H_{u12} = -2f$ as well as the plane-wave metric with $K = f^2(x_1^2 + x_2^2)$.

The same plane-wave metric may be supported by different combinations of NS-NS or R-R backgrounds, leading to different string models with different properties. For example, the metric with $K = f^2(x_1^2 + \cdots + x_8^2)$ can be supported by the 5-form background leading to the string model [13, 14, 15] with maximal supersymmetry [16], or by a combination of NS-NS and R-R 3-form fluxes giving a model with 1/2 of maximal supersymmetry.

Here we shall study in detail two superstring models which represent fundamental strings propagating in the the same plane-wave metric supported by two different — "S-dual" — backgrounds: the NS-NS 3-form field in one case, and the R-R 3-form field in another. These string models are of interest, in particular, because they are related by a special limit to the $AdS_3 \times S^3$ models with NS-NS and R-R 3-form fluxes, respectively. As was discussed in [10], starting with a string model in generic NS-NS background and applying a boost combined with a rescaling of coordinates and $\alpha' \to 0$ limit (i.e. performing a string-theory generalization of the Penrose limit [6]) one finishes with a particular plane-wave string model associated with the original background. In the case of the NS-NS $AdS_3 \times S^3$ model, i.e. SL(2, $\mathbb{R}$) $\times$ SU(2) WZW theory, this limit produces [9, 10] a plane-wave model which is a direct generalization of the NW model, i.e. the WZW theory for the group $E_2^c$ [7]. The direct light-cone gauge solution of the corresponding superstring model was previously given in [18, 19].

[1]As a result, the corresponding light-cone string spectrum (i.e. spectrum of $H = -\frac{1}{2}p^v$) is continuous. For constant $w$ this is clear, for example, from the form of the target-space Laplace operator written in momentum representation in $(u, v)$ directions: $\nabla^2 \to \partial_i^2 - (p^u)^2 w_{ij} x_i x_j - p^u p^v$.

The same limit can be applied to string models in R-R backgrounds.[2] The plane-wave model obtained as the limit of the $AdS_3 \times S^3$ R-R model is very similar to the R-R 5-form plane-wave model [13] which is the Penrose limit [17, 14] of the $AdS_5 \times S^5$ string model. Following [13, 15] here we shall determine the corresponding light-cone gauge form of the superstring action and find the string spectrum (similar final result for the spectrum was given earlier in Appendix C of [14]).

In section 2 we shall discuss the actions of the two equivalent NS-NS superstring models (which are essentially the 2-parameter generalization of the NW model), one having non-diagonal in $(u, x_i)$ target space metric but chiral 2-d structure (with the interaction term in the light-cone gauge action having the form of coupling to a constant magnetic field background) while another having the "standard" plane-wave metric. Their relation via the Penrose limit to the $AdS_3 \times S^3$ NS-NS model will be reviewed in section 3. In section 4 we shall present the light-cone spectra of the string states of these two NS-NS models.

The R-R model will be considered in section 5. We shall first determine the structure of the corresponding quadratic Green-Schwarz light-cone gauge action, and then solve the classical equations and quantize the theory in a similar way as in [13, 15]. We shall also demonstrate that the zero-mode (supergravity) part of the spectrum coincides with the corresponding part of spectrum of the NS-NS model, as expected from S-duality symmetry of type-IIB supergravity.

In section 6 we shall study the asymptotic (large light-cone energy) form of the density of states $d_{\mathcal{E}}$ of the string spectra. We shall argue that in both the R-R and the NS-NS cases the leading behaviour of $d_{\mathcal{E}}$ is the same as in flat space. This is the expected conclusion, since the dependence of the string spectrum on the curvature scale should become negligible at large level numbers.

In section 7 we shall discuss generalizations of the NS-NS models to less supersymmetric cases by adding Melvin model type twists in the two spatial 2-planes. One particular case we shall consider corresponds to compact light-cone directions, were supersymmetry is broken unless the two curvature parameters (as well as the two twist parameters) are chosen to be equal. The resulting string spectra have rather non-trivial form and may be of interest in other contexts as well. We shall note also that compactifying the light-cone directions in the R-R model reduces the number of supersymmetries by half.

Section 8 will contain some concluding remarks.

## 2. Generalized plane wave models with NS-NS 3-form background

Here we shall consider two equivalent conformal models, the bosonic parts of which are special cases of (see [6])[3]

$$L = \partial_+ u \partial_- v + K(u,x) \partial_+ u \partial_- u + 2A_i(u,x) \partial_+ u \partial_- x^i + 2\bar{A}_i(u,x) \partial_+ x^i \partial_- u + \partial_+ x^i \partial_- x^i \ . \quad (2.1)$$

[2] For a detailed discussion of the Penrose limit at the supergravity level in the general NS-NS/R-R background see [20, 17, 21]. See also refs. [22, 23] extending [14] to $AdS_5 \times M^5$ cases.

[3] We shall use Minkowski world-sheet coordinates with $\sigma^{\pm} = \tau \pm \sigma$, and $\partial_{\pm} = \frac{1}{2}(\partial_0 \pm \partial_1)$). The string action is $S = \frac{1}{\pi \alpha'} \int d^2\sigma \ L$. The space-time light-cone coordinates are $u = y - t, \ v = y + t$.

### 2.1 Bosonic actions

When $\bar{A}_i = 0$ this model and its direct type II superstring generalization [5] are conformally invariant provided

$$-\frac{1}{2}\partial^2 K + \partial^i \partial_u A_i = 0\,, \qquad \partial^j \mathcal{F}_{ij} = 0\,. \tag{2.2}$$

A particular case

$$K = 0\,, \qquad A_i = -\frac{1}{2}\mathcal{F}_{ij}x^j\,, \qquad \mathcal{F}_{ij} = \text{const} \tag{2.3}$$

i.e.

$$L = \partial_+ u \partial_- v + \mathcal{F}_{ij} x^i \partial_+ u \partial_- x^j + \partial_+ x^i \partial_- x^i\,, \qquad i,j = 1,\ldots,n\,, \tag{2.4}$$

can be interpreted [5] as boosted products of group spaces, or, equivalently, as spaces corresponding to WZW models for non-semisimple groups (see [24]). The NW model [7] is the simplest ($D = 4$, $n = 2$) example with $A_i = -f\epsilon_{ij}x^j$.[4]

Here the NS-NS 2-form has the non-trivial components $B_{uj} = \frac{1}{2}\mathcal{F}_{ij}x^i$, so that $H_{uij} = -\mathcal{F}_{ij}$. The corresponding Lorentz connection with torsion ($\omega_\pm = \omega \pm \frac{1}{2}H$) has the following non-zero components [5]

$$\omega_{-\hat{u}\hat{i}} = \frac{1}{2}\partial_i K du + \mathcal{F}_{ij} dx^j = \mathcal{F}_{ij} dx^j\,, \quad \omega_{+\hat{i}\hat{j}} = -\mathcal{F}_{ij} du\,, \quad \omega_{+\hat{u}\hat{i}} = \left(\frac{1}{2}\partial_i K - \partial_u A_i\right) du = 0\,. \tag{2.5}$$

Another conformal model we shall consider is[5]

$$\begin{aligned} L &= \partial_+ u \partial_- v - w_{ij} x^i x^j \partial_+ u \partial_- u + \frac{1}{2}\mathcal{F}_{ij} x^i \left(\partial_+ u \partial_- x^j - \partial_- u \partial_+ x^j\right) + \partial_+ x^i \partial_- x^i\,, \\ w_{ii} &= \frac{1}{4}\mathcal{F}_{ij}\mathcal{F}^{ij}\,. \end{aligned} \tag{2.6}$$

Below we shall be interested in the case when

$$w_{ij} = f^2 \delta_{ij}\,, \qquad i,j = 1,\ldots,n\,, \tag{2.7}$$

where $n$ is the dimension of $x^i$ space. $n = 2$ corresponds again to the $D = 4$ NW model [7]: $\mathcal{F}_{ij} = 2f\epsilon_{ij}$.

The two models (2.4) and (2.6) have the same 3-form field and their metrics are related by a $u$-dependent coordinate transformation: $u$-dependent rotations in the independent $x^i$ planes where $\mathcal{F}_{ij}$ is non-zero (see next section and for $n = 2$ also [12, 5, 25]). When $u$ is non-compact, the two models are thus equivalent. These models have the R-R analogs where the NS-NS 3-form is replaced by the R-R one.

The models (2.4) or (2.6) are exactly solvable since in the light-cone gauge $u \sim p^u \tau$ the transverse theory becomes gaussian. The same happens also in the fermionic sector. The solution of the bosonic $n = 2$ model (i.e. the NW model) interpreted as a WZW theory was

[4]These $\sigma$-models (or corresponding non-semisimple WZW models) can be obtained [9, 10] by singular boosts and rescalings of levels from semisimple WZW models based on direct products of $\mathrm{SL}(2,\mathbb{R})_{-k}$, $\mathrm{SU}(2)_k$ and $R$ factors. The models (2.1), (2.3), can be formally related by $O(d,d;R)$ duality to the flat space model [5] in the same way as for the $D = 4$ NW model [8, 3].

[5]To check normalizations note that $R_{uu} = -\frac{1}{2}\partial^2 K = w_{ii}$, $\frac{1}{4}H_{uij}H_{uij} = \frac{1}{4}\mathcal{F}_{ij}\mathcal{F}_{ij}$.

discussed in [8]. In the general case of compact $u$ (periodic $u+v$ coordinate) it was solved explicitly in [18]. In the case of non-compact $u$ its solution in the light-cone gauge was also discussed in [25]. The supersymmetric case of NW model was discussed from WZW CFT point of view in [11] and was solved explicitly in the light-cone gauge in [19]. Below we will generalize the string spectra found in [18, 19] to the $n=4$ (i.e. two 2-plane) case, i.e. where

$$\mathcal{F}_{kl} = 2f_1\epsilon_{kl} \quad (k,l=1,2)\,, \qquad \mathcal{F}_{k'l'} = 2f_2\epsilon_{k'l'} \quad (k',l'=3,4)\,, \tag{2.8}$$

so that (2.6) becomes (we ignore free decoupled coordinates)

$$\begin{aligned} L = {} & \partial_+ u\partial_- v - \left(f_1^2 x_k^2 + f_2^2 x_{k'}^2\right)\partial_+ u\partial_- u + \partial_+ x^k\partial_- x^k + \partial_+ x^{k'}\partial_- x^{k'} + \\ & + f_1\epsilon_{kl}x^k\left(\partial_+ u\partial_- x^l - \partial_- u\partial_+ x^l\right) + f_2\epsilon_{k'l'}x^{k'}\left(\partial_+ u\partial_- x^{l'} - \partial_- u\partial_+ x^{l'}\right). \end{aligned} \tag{2.9}$$

The $n=4$ model we shall be interested in is related by a limit to $AdS_3\times S^3$ NS-NS model where $\mathcal{F}_{ij}$ is the $4\times 4$ self-dual constant matrix, i.e.

$$f_1 = f_2 = f\,, \tag{2.10}$$

so that (2.9) takes the following explicit form

$$\begin{aligned} L = {} & \partial_+ u\partial_- v - f^2\left(x_k^2 + x_{k'}^2\right)\partial_+ u\partial_- u + \partial_+ x^k\partial_- x^k + \partial_+ x^{k'}\partial_- x^{k'} + \\ & + f\left[\epsilon_{ij}x^i\left(\partial_+ u\partial_- x^j - \partial_- u\partial_+ x^j\right) + \epsilon_{i'j'}x^{i'}\left(\partial_+ u\partial_- x^{j'} - \partial_- u\partial_+ x^{j'}\right)\right]. \end{aligned} \tag{2.11}$$

### 2.2 Fermionic actions

Let us now discuss the fermionic terms in the corresponding sigma model action. The fermionic part of (1,1) supersymmetric model corresponding to (2.4) is [5, 18, 19][6]

$$L_F = \psi_L^{\hat u}\partial_-\psi_L^{\hat v} + \psi_L^i\partial_-\psi_L^i + \mathcal{F}_{ij}\partial_- x^j\psi_L^{\hat u}\psi_L^i + \psi_R^{\hat u}\partial_+\psi_R^{\hat v} + \psi_R^i\partial_+\psi_R^i - \mathcal{F}_{ij}\partial_+ u\psi_R^i\psi_R^j\,. \tag{2.12}$$

In the light-cone gauge

$$u = 2\alpha' p^u\tau\,, \qquad \psi^u_{L,R} = 0 \tag{2.13}$$

it becomes

$$L_F = \psi_L^i\partial_-\psi_L^i + \psi_R^i\left(\delta_{ij}\partial_+ - \alpha' p^u\mathcal{F}_{ij}\right)\psi_R^j\,. \tag{2.14}$$

The coupling to $\mathcal{F}_{ij}$ here is similar to a coupling of a superparticle to a magnetic field.

The analogous light-cone gauge action corresponding to the model (2.6) has "magnetic" couplings in the both left and right sectors, with the opposite signs (reflecting opposite $H_3$ contributions)

$$L_F = \psi_L^i\left(\delta_{ij}\partial_- + \frac{1}{2}\alpha' p^u\mathcal{F}_{ij}\right)\psi_L^i + \psi_R^i\left(\delta_{ij}\partial_+ - \frac{1}{2}\alpha' p^u\mathcal{F}_{ij}\right)\psi_R^i\,. \tag{2.15}$$

Like the bosonic actions, the two fermionic actions (2.14) and (2.15) are related by coordinate rotations in each eigen-plane of $\mathcal{F}_{ij}$ with phase proportional to $f_k u$.

[6] The hatted indices are tangent-space indices, with only $\psi^{\hat v}$ being different from $\psi^v$.

The corresponding fermionic actions in the Green-Schwarz approach are also quadratic if one chooses the light-cone gauge

$$u = 2\alpha' p^u \tau\,, \qquad \Gamma^u \theta^I = 0\,. \tag{2.16}$$

Then the only coupling to the background is through the spinor covariant derivatives $(d + \frac{1}{4}\omega_{\pm \hat{m}\hat{n}}\Gamma^{\hat{m}\hat{n}})$. In the case of the background corresponding to (2.4)

$$D_+\theta = d\theta - \frac{1}{4}\mathcal{F}_{ij}\Gamma^{ij} du\theta\,, \qquad D_-\theta = d\theta + \frac{1}{4}\mathcal{F}_{ij}\Gamma^{ui} dx^j\theta\,, \tag{2.17}$$

so that in the light-cone gauge one is left with the quadratic action containing one extra term compared to the flat space GS action (cf. (2.14))

$$L_F = iS_L\partial_- S_L + iS_R\left(\partial_+ - \frac{1}{4}\alpha' p^u \mathcal{F}_{ij}\gamma^{ij}\right)S_R\,. \tag{2.18}$$

Here $S_L, S_R$ are the SO(8) spinors. This action has the same structure as (2.14), found in the NSR approach.

The light-cone GS action corresponding to the model (2.6) has, like (2.15), both left and right fermions coupled to $p^u\mathcal{F}_{ij}$ (here $\omega_{\pm ij} = \mp\frac{1}{2}\mathcal{F}_{ij}du$)

$$L_F = iS_L\left(\partial_- + \frac{1}{8}\alpha' p^u \mathcal{F}_{ij}\gamma^{ij}\right)S_L + iS_R\left(\partial_+ - \frac{1}{8}\alpha' p^u \mathcal{F}_{ij}\gamma^{ij}\right)S_R\,. \tag{2.19}$$

Again, the two actions (2.18) and (2.19) are related by a local Lorentz rotation of the space-time fermions.

### 2.3 Supersymmetry

Let us comment on supersymmetry of these models. As is well known, pure plane-wave background (with $H_{\mu\nu\rho} = 0$) preserves 1/2 of maximal space-time supersymmetry (see, e.g., [26, 27]).[7] In the presence of a $H_{uij} = -\mathcal{F}_{ij}$, the form of the light-cone Green-Schwarz action (2.19) corresponding to (2.6) (or (2.18) corresponding to (2.4)) implies that the Killing spinor equations are $(\partial_u \mp \frac{1}{8}\mathcal{F}_{ij}\Gamma^{ij})\theta_{L,R} = 0, \partial_i\theta_{L,R} = 0$, with $\Gamma^u\theta_{L,R} = 0$. When $u$ is non-compact as in the case we are interested in here, one can always solve these equations for any $\mathcal{F}_{ij}$ and find a consistent Killing spinor. Thus the amount of supersymmetry is the same (i.e. 1/2) as in the absence of $H_3$ background (see [5, 27]).

If $u$ is a compact coordinate (as in the "magnetic" models of [18, 19], see also section 7), then the models (2.4), (2.18) and (2.6), (2.19) are no longer equivalent (for generic values of the background field parameters) and have different amounts of space-time supersymmetry. In the case of the "chiral" model (2.4), (2.18) considered in [5, 18, 19] constant shifts of SO(8) spinor $S_L$ are always a symmetry, while there is no symmetry in $S_R$ sector

[7] The existence of a global supersymmetry (i.e. the analog of the flat space invariance of the GS action under $\theta \to \theta + \epsilon$, $e^{\hat{\mu}} \to e^{\hat{\mu}} + i\bar{\epsilon}^I\Gamma^{\hat{\mu}}D\theta^I, e^{\hat{\mu}} = e^{\hat{\mu}}_\nu dx^\nu$) is the same as the Killing spinor condition $D\epsilon = 0$, i.e. $(\partial_u - \frac{1}{2}\sum_i f_i x_i \Gamma^{iu})\epsilon^I = 0,\ \partial_i\epsilon^I = 0$. It has solution $\epsilon = \epsilon_0$=const, where $\Gamma^u\epsilon_0 = 0$, i.e. we get 1/2 of maximal supersymmetries preserved.

for generic $\mathcal{F}_{ij}$. This model will thus have additional supersymmetry in the $S_R$ sector, i.e. the two-parameter generalization of the constant magnetic model of [18] will preserve 8+4 real supersymmetries in the $f_1 = f_2$ case (analogous mechanism is at work in the supersymmetric magnetic models of [28]).

In the case of the model (2.6), (2.19) with compact $u$ all supersymmetry will be again broken if $\mathcal{F}_{ij}$ is generic. However, fractions of supersymmetry may be preserved for special choices of $\mathcal{F}_{ij}$. For example, for a self-dual $\mathcal{F}_{ij}$, i.e. in the case of $f_1 = f_2$, there will be constant Killing spinors in both $S_L$ and $S_R$ sectors of (2.19) provided they satisfy $(\gamma_{12} + \gamma_{34})S_{L.R} = 0$, i.e. $\gamma_{1234}S_{L,R} = S_{L,R}$. Thus (2.19) will preserve 1/4 of the maximal number of supersymmetries, i.e. will have 8 real supercharges.

## 3. "Plane-wave" limit of $AdS_3 \times S^3$ string sigma model

The model (2.9) arises as a limit of $AdS_3 \times S^3$ with 3-form flux or $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ WZW model following the limiting procedure described in [10, 17]. This limit can be taken directly in the string sigma model action (which remains invariant under the corresponding rescaling of coordinates and parameters) or separately in the metric and $B_{\mu\nu}$. Let us start with the following parametrization of the metric of $AdS_3 \times S^3$

$$ds^2 = f^2 \left[ -\left(1 + r_1^2\right) dt^2 + \frac{dr_1^2}{1 + r_1^2} + r_1^2 d\varphi_1^2 + \left(1 - r_2^2\right) d\psi^2 + \frac{dr_2^2}{1 - r_2^2} + r_2^2 d\varphi_2^2 \right] . \qquad (3.1)$$

Let us rescale $r_1, r_2$ by $\epsilon$, $f \to f/\epsilon$, set

$$t = u\,, \qquad \psi = u + \frac{1}{2}\epsilon^2 v\,, \qquad (3.2)$$

and consider the limit $\epsilon \to 0$.[8] This gives us the plane-wave metric:

$$ds^2 = dudv - f^2 x_i^2 du^2 + dx^i dx^i\,, \qquad i = 1, \dots, 4\,, \qquad (3.3)$$

where we have introduced cartesian coordinates $x^i$,

$$dx^i dx^i = dr_1^2 + r_1^2 \varphi_1^2 + dr_2^2 + r_2^2 d\varphi_2^2\,,$$

and made a finite rescaling of $v$ and $x^i$ by powers of $f$. The components of $B_{\mu\nu}$ are found by the same limiting procedure, with the overall rescaling compensated by rescaling of $f$.

The final result is the plane-wave sigma model action (2.11). Note that if $t$ is non-compact, both $u$ and $v$ are non-compact in the limit $\epsilon \to 0$: the period of $v$ which is equal to $8\pi/\epsilon^2$ becomes infinite.

For generality, in what follows we shall keep $f_1, f_2$ parameters arbitrary. The case of $f_1 = f_2$ corresponding to the limit of $AdS_3 \times S^3$ has SO(4) invariant metric, but this symmetry is in any case broken down to $\mathrm{SO}(2) \times \mathrm{SO}(2)$ by the antisymmetric tensor terms.

[8] The rescaling of the size of the space $f^2 = R^2/\alpha'$ is equivalent to $\alpha' \to 0$, explaining why the resulting background solves string equations to all orders in $\alpha'$.



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The model (2.9) is equivalent to

$$L = \partial_+ u \partial_- v + 2 f_1 \epsilon_{kl} x^k \partial_+ u \partial_- x^l + 2 f_2 \epsilon_{k'l'} x^{k'} \partial_+ u \partial_- x^{l'} + \partial_+ x^k \partial_- x^k + \partial_+ x^{k'} \partial_- x^{k'} \,. \quad (3.4)$$

The lagrangian (3.4) is a particular case of (2.4) with block-diagonal $4 \times 4$ matrix $\mathcal{F}_{ij}$. In polar coordinates in the two planes it takes the form

$$\begin{aligned} L = {} & \partial_+ u \partial_- v + 2 f_1 r_1^2 \partial_+ u \partial_- \varphi_1 + 2 f_2 r_2^2 \partial_+ u \partial_- \varphi_2 + \\ & + \partial_+ r_1 \partial_- r_1 + r_1^2 \partial_+ \varphi_1 \partial_- \varphi_1 + \partial_+ r_2 \partial_- r_2 + r_2^2 \partial_+ \varphi_2 \partial_- \varphi_2 \,. \end{aligned} \quad (3.5)$$

The model (2.9) (corresponding to the metric (3.3) for $f_1 = f_2$) is related to (3.5) by the sigma model field redefinition[9]

$$\varphi_1 = \varphi_1' - f_1 u \,, \qquad \varphi_2 = \varphi_2' - f_2 u \,. \quad (3.6)$$

Indeed, it transforms (3.5) into

$$\begin{aligned} L = {} & \partial_+ u \partial_- v - \left( f_1^2 r_1^2 + f_2^2 r_2^2 \right) \partial_+ u \partial_- u + \partial_+ r_1 \partial_- r_1 + r_1^2 \partial_+ \varphi_1' \partial_- \varphi_1' + \partial_+ r_2 \partial_- r_2 + \\ & + r_2^2 \partial_+ \varphi_2' \partial_- \varphi_2' + f_1 r_1^2 (\partial_+ u \partial_- \varphi_1' - \partial_- u \partial_+ \varphi_1') + f_2 r_2^2 (\partial_+ u \partial_- \varphi_2' - \partial_- u \partial_+ \varphi_2'), \end{aligned} \quad (3.7)$$

which is equivalent to (2.9).

The fermions are included in a straightforward way, according to (2.15) for (2.9) and (2.14) for (3.4).

## 4. Solution of the NS-NS plane wave superstring model

Below we shall first discuss the spectrum of the two models (3.4) and (2.9). We shall only consider the case of $y = \frac{1}{2}(u + v)$ being non-compact when they are equivalent (cf. (3.6)) and are related to the plane-wave limit of the $AdS_3 \times S^3$ theory. In the light-cone gauge they are simply direct products of the NW model discussed in [18, 25], so the spectrum is readily obtained from the previous results.

The conformal model (3.4) can be solved by a simple extension of the single 2-plane case ($f_2 = 0$) studied in [18] for the bosonic case, and [19] for the supersymmetric case. Let us set (note that t is different from $t$ in (3.2))

$$u = y - \mathrm{t} \,, \qquad v = y + \mathrm{t} \,. \quad (4.1)$$

In [18, 19], the coordinate $y$ was compact describing a circle of radius $R$. In the limit $R \to \infty$ we are interested in here the charges $Q_L$, $Q_R$ of [19] become equal (only zero winding sector contributes) and are replaced by the momentum operator $p_y$ with continuous spectrum.

[9]From the target space point of view this is a coordinate transformation combined with the gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + 2\partial_{[\mu}\Lambda_{\nu]}$, $\Lambda_i = u f \epsilon_{ij} x^j$. Note that this coordinate transformation is globally defined as long as $u$ is non-compact. If $u$ is compact, a similar redefinition in the two planes corresponds to twists considered in connection with "rotating" NS5-NS1 system (related to rotating 5-d black holes [29]) in [30] and also (in the case when $u$ is replaced by a space-like coordinate) in connection with the Melvin-type "magnetic" models in [28].



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The generalization of the superstring expressions in [18, 19] is then as follows. Let $\hat{N}_R$ and $\hat{N}_L$ denote number of states operators, which have the standard form as in free superstring theory (in the NSR approach) $\hat{N}_{R,L} = N_{R,L} - a\,, a^{(\mathrm{R})} = 0,\ a^{(\mathrm{NS})} = 1/2$ (explicit expressions can be found in [19, 31]). Let us introduce the angular momentum operators $\hat{J}_1 \equiv \hat{J}_{12}$ and $\hat{J}_2 \equiv \hat{J}_{34}$, which generate rotations in the respective 2-planes (i.e. generate shifts in $\varphi_1$ and $\varphi_2$). They contain contributions of both bosonic and fermionic oscillators for the respective coordinates (of the same form as in flat space) and can be written as follows:

$$\hat{J}_s = \hat{J}_{sL} + \hat{J}_{sR}\,, \quad \hat{J}_{sL} = l_{sL} + \frac{1}{2} + s_{sL}\,, \quad \hat{J}_{sR} = -l_{sR} - \frac{1}{2} + s_{sR}\,, \qquad s = 1,2\,, \tag{4.2}$$

where the orbital momenta in each plane $l_{L,R} = 0,1,2,\ldots$ are related to the Landau quantum number $l$ and radial quantum number $k$ by $l = l_L - l_R$ and $2k = l_L + l_R - |l|$, and $s_{R,L}$ are the spin components. For the type II superstring theory, the spectrum for $R \to \infty$ directly following from [19] may be written as

$$E^2 - p_\alpha^2 = \frac{2}{\alpha'}\left(\hat{N}_L + \hat{N}_R\right) + p_y^2 - 4\,(p_y + E)\left(f_1 \hat{J}_{1R} + f_2 \hat{J}_{2R}\right). \tag{4.3}$$

Here $\hat{N}_R - \hat{N}_L = 0$ and the operators $N_{L,R}$ and $J_{sL,R}$ have the standard flat-space form in terms of the bosonic and fermionic creation/annihilation operators for the corresponding coordinates. $p_\alpha$ are momenta in additional 4 spectator directions, and $E$ is conjugate to the time coordinate t.

As expected on the basis of the supersymmetry, the above spectrum is tachyon free. This can be seen in a manifest way by writing (4.3) in the form

$$\left[E + 2\left(f_1 \hat{J}_{1R} + f_2 \hat{J}_{2R}\right)\right]^2 = p_\alpha^2 + \frac{2}{\alpha'}\left(\hat{N}_L + \hat{N}_R\right) + \left[p_y - 2\left(f_1 \hat{J}_{1R} + f_2 \hat{J}_{2R}\right)\right]^2, \tag{4.4}$$

which shows that there is no state for which $E$ has an imaginary component (note that $\hat{N}_L, \hat{N}_R = 0,1,2,\ldots$).

In the present context, it is natural to use the light-cone energy $H = -p_u$, conjugate to the coordinate $u$ (which is actually the original time of $AdS_3$, see (3.2)). We have the relations

$$p_u = \frac{1}{2}(p_y - E)\,, \qquad p_v = \frac{1}{2}(p_y + E)\,, \qquad p^u = 2p_v\,, \qquad p^v = 2p_u\,. \tag{4.5}$$

In terms of $p_u, p_v$ the light-cone energy spectrum takes the form

$$H = -p_u = \frac{p_\alpha^2}{4p_v} + \frac{1}{2\alpha' p_v}\left(\hat{N}_L + \hat{N}_R\right) - 2\left(f_1 \hat{J}_{1R} + f_2 \hat{J}_{2R}\right), \tag{4.6}$$

i.e.

$$H = \frac{p_\alpha^2}{4p_v} + \frac{1}{2\alpha' p_v}\mathcal{H}\,, \qquad \mathcal{H} = \hat{N}_L + \hat{N}_R - 2\left(\mathrm{m}_1 \hat{J}_{1R} + \mathrm{m}_2 \hat{J}_{2R}\right), \quad \mathrm{m}_{1,2} \equiv 2\alpha' p_v f_{1,2}\,. \tag{4.7}$$

It has the same form of the Landau-type spectrum of a particle or open string in a constant magnetic field (see, e.g., [32] for a review).

The partition function vanishes by virtue of supersymmetry of the plane-wave background.[10]

The equivalent spectrum can be found also starting with (2.9), i.e. the same model written in the rotated coordinate system (3.6). It is convenient to introduce the complex coordinates $z_1 = x_1 + ix_2 = r_1 e^{i\varphi_1'}$ and $z_2 = x_3 + ix_4 = r_2 e^{i\varphi_2'}$. Then (2.9) becomes

$$L = \partial_+ u \partial_- v + \sum_{s=1,2} \Big( -f_s^2 z_s z_s^* \partial_+ u \partial_- u + \partial_+ z_s \partial_- z_s^* + \\ + \frac{1}{2} i f_s \left[ \partial_+ u \left( z_s \partial_- z_s^* - z_s^* \partial_- z_s \right) - \partial_- u ( z_s \partial_+ z_s^* - z_s^* \partial_+ z_s ) \right] \Big) . \quad (4.8)$$

The solution to the equations of motion can be easily found, e.g., by using the solutions of the model (3.4) (explicitly given in [18, 19]) and performing the coordinate transformation (3.6) . In spite of the apparent mass term for the bosonic fields $z_1, z_2$, the theory is again solved in terms of massless free oscillators. Since the equation of motion for $u$ is $\partial_+ \partial_- u = 0$ we fix the light-one gauge $u = 2\alpha' p^u \tau = 4\alpha' p_v \tau$. Then the equations for $z_1, z_2$ become

$$\partial_+ \partial_- z_s + \mathrm{m}_s^2 z_s + i \mathrm{m}_s (\partial_- z_s - \partial_+ z_s) = 0 , \qquad \mathrm{m}_s \equiv 2\alpha' p_v f_s , \quad s = 1, 2 , \quad (4.9)$$

and thus are solved by

$$z_s = e^{-2i\mathrm{m}_s \sigma} Z_s , \qquad Z_s = Z_{s+}(\sigma_+) + Z_{s-}(\sigma_-) . \quad (4.10)$$

Because $z_{1,2}(\sigma + \pi) = z_{1,2}(\sigma)$, the free fields $Z_{1,2}$ obey the twisted boundary conditions

$$Z_1(\sigma + \pi) = e^{2i\mathrm{m}_1 \pi} Z_1(\sigma) , \qquad Z_2(\sigma + \pi) = e^{2i\mathrm{m}_2 \pi} Z_2(\sigma) . \quad (4.11)$$

The mass spectrum is then found in the canonical way, by computing the Virasoro operators and imposing the usual Virasoro conditions for physical states (which are explicitly solved in the light-cone gauge).

A shortcut to the explicit form of the spectrum is to relate the energy operator in the coordinate system corresponding to (3.7) to the energy operator (4.3) corresponding to (3.4). From the coordinate transformation (3.6) (with $u' = u,\ v' = v$), we learn that $\hat{J}_s' = -i \frac{\partial}{\partial \varphi_s'} = -i \frac{\partial}{\partial \varphi_s} = \hat{J}_s$ and

$$E' = i \frac{\partial}{\partial t'} = i \frac{\partial}{\partial t} - i f_1 \frac{\partial}{\partial \varphi_1} - i f_2 \frac{\partial}{\partial \varphi_2} = E + f_1 \hat{J}_1 + f_2 \hat{J}_2 , \quad (4.12)$$

$$p_y' = -i \frac{\partial}{\partial y'} = -i \frac{\partial}{\partial y} + i f_1 \frac{\partial}{\partial \varphi_1} + i f_2 \frac{\partial}{\partial \varphi_2} = p_y - f_1 \hat{J}_1 - f_2 \hat{J}_2 . \quad (4.13)$$

Hence $p_u' = p_u - f_1 \hat{J}_1 - f_2 \hat{J}_2$, while $p_v$ does not change, $p_v' = p_v = \frac{1}{2}(p_y' + E') = \frac{1}{2}(p_y + E)$. Thus the spectrum is given by (4.3) expressed in terms of $E'$, $p_y'$:

$$E'^2 - p_\alpha^2 = \frac{2}{\alpha'} \left( \hat{N}_L + \hat{N}_R \right) + {p_y'}^2 - 2 \left( p_y' + E' \right) \left[ f_1 \left( \hat{J}_{1R} - \hat{J}_{1L} \right) + f_2 \left( \hat{J}_{2R} - \hat{J}_{2L} \right) \right] , \quad (4.14)$$

[10]In the bosonic theory, in the limit $R \to \infty$ partition function becomes the same as the flat space partition function.

supplemented by $\hat{N}_R - \hat{N}_L = 0$. Equivalently, the analog of (4.6) is

$$H \equiv -p'_u = \frac{p_\alpha^2}{4p_v} + \frac{1}{2\alpha' p_v}\left(\hat{N}_L + \hat{N}_R\right) - f_1\left(\hat{J}_{1R} - \hat{J}_{1L}\right) - f_2\left(\hat{J}_{2R} - \hat{J}_{2L}\right), \tag{4.15}$$

i.e. (cf. (4.7))

$$H = \frac{p_\alpha^2}{4p_v} + \frac{1}{2\alpha' p_v}\mathcal{H}, \qquad \mathcal{H} = \hat{N}_L + \hat{N}_R - \mathrm{m}_1\left(\hat{J}_{1R} - \hat{J}_{1L}\right) - \mathrm{m}_2\left(\hat{J}_{2R} - \hat{J}_{2L}\right). \tag{4.16}$$

For $f_1 = f_2 = f$ this becomes simply

$$\begin{aligned} H &= \frac{p_\alpha^2}{4p_v} + \frac{1}{2\alpha' p_v}\left(\hat{N}_L + \hat{N}_R\right) - f\left(\hat{J}_R - \hat{J}_L\right), \\ &\hat{J}_R \equiv \hat{J}_{1R} + \hat{J}_{2R}, \qquad \hat{J}_L \equiv \hat{J}_{1L} + \hat{J}_{2L}, \end{aligned} \tag{4.17}$$

i.e. is related to $H$ in (4.6) (with $f_1 = f_2$) by the shift $H \to H + f(\hat{J}_R + \hat{J}_L)$. This spectrum is equivalent to the spectrum given in appendix C of [14] (which appeared while this paper was in preparation) with a proper identification of modes.

Note that in spite of the presence of an apparent bosonic mass term in (4.8) the spectrum has again the same "magnetic" form as in (4.6).

String theory in these magnetic backgrounds is periodic under $\mathrm{m}_1 \to \mathrm{m}_1 + 2n_1$, $\mathrm{m}_2 \to \mathrm{m}_2 + 2n_2$, with integer $n_1, n_2$ (the periodicity in the case $\mathrm{m}_2 = 0$ is discussed in [18, 19]). Under such shifts, the spectrum remains the same after relabelling the mode operators. This is clear from (4.16): the angular momentum components $\hat{J}_{1R}$, etc., appearing in this expression have integer and half-integer eigenvalues. Shifts of $\mathrm{m}_1, \mathrm{m}_2$ by even integers can be absorbed into integer shifts of $\hat{N}_R$, $\hat{N}_L$ which in turn can be produced by a relabelling of mode operators (that shifts the vacuum energy).

The case of $\mathrm{m}_1 = \mathrm{m}_2 = \mathrm{m}$ is special: here the theory becomes periodic under $\mathrm{m} \to \mathrm{m} + n$, with $n$ being any integer number. This can be understood by noting that the eigenvalues of $\hat{J}_R$ are always integer, so that a shift $\mathrm{m} \to \mathrm{m} + n$ can be absorbed into an integer shift of $\hat{N}_R, \hat{N}_L$. This periodicity follows also directly from the form of the string action. Consider in particular the "left" term in (2.19), i.e. $\sim S_L[\partial_0 - \partial_1 + \mathrm{m}(\gamma_{12} + \gamma_{34})]S_L$. Since $\gamma_{12} + \gamma_{34} = 2\gamma_{12}P$, $P \equiv \frac{1}{2}(I - \gamma_{1234})$, by splitting $S_L$ into two parts using the projector $P$ (see also section 5.2), we conclude that the $\sigma$-dependence of the solution for the $PS_L = S_L$ part is given by $\exp(2\mathrm{m}\sigma\gamma^{12})$. The boundary condition for the GS fermions is the periodicity under $\sigma \to \sigma + \pi$; this requirement is invariant under $\mathrm{m} \to \mathrm{m} + 1$, determining the periodicity of the spectrum. In particular, the string theory with $\mathrm{m}_1 = \mathrm{m}_2 = 1$ is equivalent to the theory with $\mathrm{m}_1 = \mathrm{m}_2 = 0$ (i.e. to flat space). Without loss of generality, the values of m can thus be restricted to the interval $0 \leq \mathrm{m} < 1$.

## 5. Superstring model for plane-wave R-R 3-form background

### 5.1 Lagrangian

The S-dual to the above NS-NS background has the same plane wave metric corresponding to the limit of the $AdS_3 \times S^3$ now supported by the R-R $F_3$ flux of the same null structure as $H_3$ (cf. (2.8))

$$ds^2 = dudv - f^2 x_i^2 du^2 + dx_i^2 \,, \tag{5.1}$$

$$F_{uij} = -\mathcal{F}_{ij} \,, \mathcal{F}_{12} = \mathcal{F}_{34} = 2f \,. \tag{5.2}$$

In general, it is possible to argue that given the above plane-wave metric supported by the R-R background $F_{ui_1,\ldots,i_p} \sim \mathcal{F}_{i_1,\ldots,i_p}$ =const, the only non-zero fermionic contribution to the type-IIB superstring action in the $\kappa$-symmetry light-cone gauge $\Gamma^u\theta^I = 0$ is quadratic in fermions and comes from the direct generalization of the fermion "kinetic" term in the flat-space GS action

$$i\left(\eta^{ab}\delta_{IJ} - \epsilon^{ab}\rho_{3IJ}\right)\partial_a x^m \bar{\theta}^I \Gamma_m \left(\hat{D}_b\right)^{JK}\theta^K \,, \tag{5.3}$$

where $\theta^I$ are MW spinors ($I = 1, 2$) and $\rho_3 = \text{diag}(1, -1)$. The derivative $\hat{D}_b$ is the generalized covariant derivative that appears in the Killing spinor equation (or gravitino transformation law) in type-IIB supergravity [33]. Acting on the real MW spinors $\theta^I$ it has the form[11]

$$\hat{D}_a = \partial_a + \frac{1}{4}\partial_a x^k \left[\left(\omega_{mnk} - \frac{1}{2}H_{mnk}\rho_3\right)\Gamma^{mn} - \left(\frac{1}{3!}F_{mnl}\Gamma^{mnl}\rho_1 + \frac{1}{2\cdot 5!}F_{mnlpq}\Gamma^{mnlpq}\rho_0\right)\Gamma_k\right]. \tag{5.4}$$

Here $\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\rho_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ act in the $I, J$ space.

The non-zero contribution to the action comes only from this quadratic covariant derivative term where both $\partial_a x^m$ factors in (5.3) are replaced by $\partial_a u$, i.e. (after fixing the bosonic light-cone gauge $u = 2\alpha' p^u \tau$) by $2\alpha' p^u \delta_u^m \delta_a^0$. In the flat space light-cone gauge GS action the spinors $\theta^1 \equiv \theta_R$ and $\theta^2 \equiv \theta_L$ become the left and right moving 2-d fermions. The same happens in the plane-wave background, and the surviving fermionic interaction terms are proportional to

$$\left(\theta_L \Gamma^v \Gamma^{ij}\theta_L - \theta_R \Gamma^v \Gamma^{ij}\theta_R\right)\mathcal{F}_{ij}$$

in the $H_3$ case ($\rho_3$ is diagonal), and to

$$\theta_L \Gamma^v \Gamma^{i_1,\ldots,j_s}\theta_R \mathcal{F}_{i_1,\ldots,j_s}$$

in the $F_3$ and $F_5$ cases ($\rho_1$ and $\rho_0$ are off-diagonal). In the $F_5$ case we reproduce the result [13] of the direct derivation of the GS action in the maximally supersymmetric plane-wave background of [16] (see also [15]).

As expected, the fermionic interaction term has chiral (in 2-d sense) structure in the NS-NS case, but the non-chiral, "mass-term" structure in the R-R case. This leads to important differences in the properties (in particular, the spectrum) of the NS-NS and the R-R models.

Explicitly, in the light-cone gauge[12]

$$u = 2\alpha' p^u \tau \,, \qquad \gamma^u \theta^I = 0 \,, \tag{5.5}$$

[11] We set to zero all background fields except $H_3$, $F_3$ and $F_5$.

[12] In what follows we shall use the 16-component spinors and $16 \times 16$ $\gamma$-matrices instead of 32-component spinors and $32 \times 32$ $\Gamma$-matrices used above. Note that $\Gamma^u = \Gamma^9 - \Gamma^0$, $\Gamma^v = \Gamma^9 + \Gamma^0$, $\Gamma^u\Gamma_u = 1$.

where $\theta^I = (\theta_R, \theta_L)$ are 16-component MW spinors, we finish with[13]

$$L = L_B + L_F \, , \qquad L_B = \partial_+ u \partial_- v - \mathrm{m}^2 x_i^2 + \partial_+ x_i \partial_- x_i + \partial_+ x_\alpha \partial_- x_\alpha \, , \tag{5.6}$$

$$L_F = i\theta_R \gamma^v \partial_+ \theta_R + i\theta_L \gamma^v \partial_- \theta_L - 2i\mathrm{m}\theta_L \gamma^v M \theta_R \, , \tag{5.7}$$

$$\mathrm{m} \equiv \alpha' p^u f = 2\alpha' p_v f \, , \tag{5.8}$$

$$M \equiv -\frac{1}{8f} \mathcal{F}_{ij} \gamma^{ij} = -\frac{1}{2} \left( \gamma^{12} + \gamma^{34} \right) \, . \tag{5.9}$$

We have absorbed one power of $p^u$ in (5.7) by the rescaling $\theta \to \sqrt{p^u}\theta$. While the $x_i$-part of the bosonic action has SO(4) symmetry, the corresponding symmetry of the fermionic action is only SO(2) $\times$ SO(2), i.e. is the same as the symmetry of the $F_3$ background.

Comparing the fermionic action (5.7) of the R-R model to the corresponding action (2.19) of the NS-NS model we see that indeed in the R-R case the coupling to the 3-form background produces a 2-d "mass" term mixing the left and right components of the fermions, while in the NS-NS case these components were decoupled. In the NS-NS case the coupling to the 3-form field changes boundary conditions for the effective free fermion fields (i.e. $S'_{L,R} = \exp[\pm \frac{1}{8} \alpha' p^u \mathcal{F}_{ij} \gamma^{ij} \sigma^\mp] S_{L,R}$). In the general case of both NS-NS and R-R 3-form backgrounds being non-zero, the GS fermionic action following from (5.3), (5.4) will contain the covariant derivatives on the left and right fermion fields as well as a mass term mixing them (this case was already briefly discussed in [14, appendix C]).

The fermionic action (5.7) is essentially the same as found in [13] in the 5-form background case. As in the action of [13] the fermionic interaction plays the role of the 2-d fermion mass term, mixing the left and right modes. Note, however, that here

$$M^2 = -\frac{1}{2} \left( 1 - \gamma^{1234} \right) \tag{5.10}$$

is (minus) a projector, in contrast to the 5-form case [13] where the corresponding mass operator had its square proportional to 1. Thus only half of the fermions are getting mass. The 4 massive + 4 massless bosons are supplemented then by the 4 massive (2-d Majorana) fermions and 4 massless fermions. As in the flat GS action and the action of [13] here the light-cone gauge GS action happens to have an effective 2-d supersymmetry.

The amount of the space-time supersymmetry here is the same as in the "S-dual" NS-NS case. The $AdS_3 \times S^3$ background with the R-R 3-form flux preserves 1/2 of the 32 supersymmetries, and its plane wave limit also has 16 supersymmetries. Note that the restriction $f_1 = f_2$ is not necessary for the 1/2 of maximal supersymmetry of the R-R plane wave model, but such model is a limit of the corresponding $AdS_3 \times S^3$ model where the equality of the radii of the two 3-dimensional factors is crucial for the supersymmetry (and also for the dilaton to be constant). The more general R-R plane-wave model with field strengths $F_{u12} = -2f_1$, $F_{u34} = -2f_2$ can be solved in a similar way as its $f_1 = f_2$ case considered below.

[13]Here $\alpha = 5, \ldots, 8$. We assume that all $\gamma$-matrices are real and symmetric, so that $M$ and $\gamma^v M$ are antisymmetric.

### 5.2 Solution of the string model

The solution and quantization of this model is similar to the one described in the R-R 5-form background case in [13, 15].

The equations of motion for $x_i$ and the fermions following from (5.6), (5.7) are

$$\partial_+\partial_- x_i + \mathrm{m}^2 x_i = 0\,, \qquad \partial_+\partial_- x_\alpha = 0\,, \tag{5.11}$$

$$\partial_+\theta_R - \mathrm{m}M\theta_L = 0\,, \qquad \partial_-\theta_L - \mathrm{m}M\theta_R = 0\,. \tag{5.12}$$

As in flat space case, let us solve the light-cone gauge condition on fermions (5.5) explicitly, expressing $\theta_{L,R}$ in terms of 8+8 independent real fermions $S_{L,R}$ (spinors of SO(8)). One can choose the following representation for the $16\times 16$ gamma matrices:

$$\gamma^u \to I\otimes\sigma^+\,, \qquad \gamma^v \to I\otimes\sigma^-\,, \qquad \gamma^i \to \gamma^i\otimes\sigma^3\,,$$

where the new $\gamma^i$'s are $8\times 8$ gamma matrices of SO(8) group. $\sigma$'s are Pauli matrices and $I$ is the $8\times 8$ identity matrix. In this representation one is left with the upper-component spinor $S$, and the 16-component and 8-component gamma matrices with transverse indices are directly related, i.e. $\gamma^{12}\to\gamma^{12}$, $\gamma^{34}\to\gamma^{34}$. Then (5.12) takes the same form but now in terms of unconstrained 8-component spinors. In particular, we learn that

$$\partial_+\partial_- S_{L,R} - \mathrm{m}^2 M^2 S_{L,R} = 0\,, \tag{5.13}$$

where $M$ now is $8\times 8$ matrix of the same structure as in (5.9).

Let us further split the fermions in the 8→4+4 way $S_L \to (S_L,\hat S_L)$, $S_R\to(S_R,\hat S_R)$, so that

$$\gamma^{1234}\begin{pmatrix} S_{L,R}\\ \hat S_{L,R}\end{pmatrix} = \begin{pmatrix} -S_{L,R}\\ \hat S_{L,R}\end{pmatrix}. \tag{5.14}$$

One may solve the constraints in (5.14) explicitly by reducing the spinors to independent 4-component ones. For example, diagonalizing $\gamma^{12}$ so that

$$\gamma^{12}\begin{pmatrix} S_{L,R}\\ \hat S_{L,R}\end{pmatrix} = -\begin{pmatrix} \Lambda S_{L,R}\\ \Sigma\hat S_{L,R}\end{pmatrix}, \qquad \Lambda^2=\Sigma^2=-I\,, \tag{5.15}$$

where $\Lambda$ and $\Sigma$ are $4\times 4$ traceless antisymmetric matrices with eigenvalues $\pm i$. We find

$$M\begin{pmatrix} S_{L,R}\\ \hat S_{L,R}\end{pmatrix} = \begin{pmatrix} \Lambda S_{L,R}\\ 0\end{pmatrix}, \tag{5.16}$$

and eqs. (5.12) take the form

$$\partial_+ S_R - \mathrm{m}\Lambda S_L = 0\,, \qquad \partial_- S_L - \mathrm{m}\Lambda S_R = 0\,, \tag{5.17}$$

$$\partial_+\hat S_R = 0\,, \qquad \partial_-\hat S_L = 0\,. \tag{5.18}$$

We are left with 4+4 independent degrees of freedom $(S_L^a,\hat S_L^A)$, $(S_R^a,\hat S_R^A)$, where $a=1,2,3,4$ and $A=1,2,3,4$. In what follows the explicit structure of the matrix $\Lambda$ (i.e. the upper $4\times 4$ block of $\gamma^{12}$) will not be important – only it square ($=-I$) will appear in the resulting hamiltonian.



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Note that $S^a_{L,R}$ do not transform as an SO(4) spinor under rotations in (1, 2, 3, 4) directions: the presence of $\gamma^{12}$ in its definition (5.15) breaks the SO(4) Lorentz symmetry down to $\mathrm{SO}(2)\times\mathrm{SO}(2)$, with the latter being the true global symmetry of the R-R 3-form background (5.2), i.e. of the fermionic sector of the string action (5.7), and thus of the whole string theory (see [15] for a related discussion in the 5-form background case).

Thus we finish with four massive 2-d bosons $x_i$ and four massive fermions $S^a_{L,R}$ of the same mass m, as well as with four massless free bosons $x_\alpha$ and four massless fermions $\hat{S}^A_{L,R}$.[14]

The Fourier expansion for generic closed-string classical solutions $x^i$ can be written as

$$x^i(\sigma,\tau) = \frac{il}{2}\Bigg[\frac{1}{\sqrt{w_0}}\left(a^i_0 e^{-2iw_0\tau} - a^{i\dagger}_0 e^{2iw_0\tau}\right) + \qquad (5.19)$$

$$+\sum_{n=1}^{\infty}\frac{1}{\sqrt{w_n}}\left[e^{-2iw_n\tau}\left(a^i_n e^{2in\sigma}+\tilde{a}^i_n e^{-2in\sigma}\right)-e^{2iw_n\tau}\left(a^{i\dagger}_n e^{-2in\sigma}+\tilde{a}^{i\dagger}_n e^{2in\sigma}\right)\right]\Bigg],$$

where

$$w_n = \sqrt{n^2+\mathrm{m}^2}\,, \qquad w_0=\mathrm{m}\,, \qquad l\equiv\sqrt{2\alpha'}\,. \qquad (5.20)$$

Note that m and $w_n$ are dimensionless; we have introduced the string scale $l$ to make the Fourier mode coefficients dimensionless. The canonical commutation relation $[\dot{x}^i(\sigma), x^j(\sigma')] = -2\pi\alpha' i\delta(\sigma-\sigma')\delta^{ij}$ then implies as usual

$$\left[a^i_n, a^{j\dagger}_m\right] = \delta_{nm}\delta^{ij}\,, \qquad \left[\tilde{a}^i_n, \tilde{a}^{j\dagger}_n\right] = \delta_{nm}\delta^{ij}\,. \qquad (5.21)$$

In addition, we have massless boson modes $a^\alpha_n, \tilde{a}^\alpha_n$ appearing in the expansion of $x^\alpha$. For the fermions, the solutions are

$$S^a_R(\sigma,\tau) = l\Bigg[c_0\left(S^a_0 e^{-2iw_0\tau}+S^{a\dagger}_0 e^{2iw_0\tau}\right)+ \qquad (5.22)$$

$$+\sum_{n=1}^{\infty}c_n\left[e^{-2iw_n\tau}\left(S^a_n e^{2in\sigma}+\tilde{S}'^a_n e^{-2in\sigma}\right)+e^{2iw_n\tau}\left(S^{a\dagger}_n e^{-2in\sigma}+\tilde{S}'^{a\dagger}_n e^{2in\sigma}\right)\right]\Bigg],$$

$$S^a_L(\sigma,\tau) = l\Bigg[c_0\left(\tilde{S}^a_0 e^{-2iw_0\tau}+\tilde{S}^{a\dagger}_0 e^{2iw_0\tau}\right)+ \qquad (5.23)$$

$$+\sum_{n=1}^{\infty}c_n\left[e^{-2iw_n\tau}\left(\tilde{S}^a_n e^{-2in\sigma}+S'^a_n e^{2in\sigma}\right)+e^{2iw_n\tau}\left(\tilde{S}^{a\dagger}_n e^{2in\sigma}+S'^{a\dagger}_n e^{-2in\sigma}\right)\right]\Bigg],$$

where $c_0$, $c_n$ are the normalization constants to be fixed below. The primed fermion modes $S'^a_n, \tilde{S}'^a_n$ are related, according to the first-order equation (5.17), to $S^a_n, \tilde{S}^a_n$, by which gives

$$\tilde{S}'^a_n = \frac{i}{\mathrm{m}}(w_n-n)\left(\Lambda\tilde{S}_n\right)^a, \qquad \tilde{S}'^{a\dagger}_n = -\frac{i}{\mathrm{m}}(w_n-n)\left(\Lambda\tilde{S}^\dagger_n\right)^a, \qquad (5.24)$$

$$S'^a_n = \frac{i}{\mathrm{m}}(w_n-n)\left(\Lambda S_n\right)^a, \qquad S'^{a\dagger}_n = -\frac{i}{\mathrm{m}}(w_n-n)\left(\Lambda S^\dagger_n\right)^a, \qquad (5.25)$$

and

$$S^a_0 = i\left(\Lambda\tilde{S}_0\right)^a, \qquad S^{a\dagger}_0 = -i\left(\Lambda\tilde{S}^\dagger_0\right)^a. \qquad (5.26)$$

[14] In what follows we will omit the hat over $S^A$, differentiating between $S$ and $\hat{S}$ by type of their indices.

In the zero-field limit $\mathrm{m} \to 0$ (i.e. $f \to 0$) we recover the standard flat-space Fourier expansions. Imposing the canonical commutation relations

$$\left\{S_L^a(\sigma,\tau), S_L^b(\sigma',\tau)\right\} = 2\pi\alpha'\delta^{ab}\delta(\sigma-\sigma'), \qquad \left\{S_R^a(\sigma,\tau), S_R^b(\sigma',\tau)\right\} = 2\pi\alpha'\delta^{ab}\delta(\sigma-\sigma') \tag{5.27}$$

and fixing the normalization constants as

$$c_0 = \frac{1}{\sqrt{2}}, \qquad c_n = \frac{\mathrm{m}}{\sqrt{\mathrm{m}^2 + (w_n - n)^2}}, \tag{5.28}$$

we get the following anti-commutation relations

$$\left\{S_m^a, S_n^{b\dagger}\right\} = \delta^{ab}\delta_{mn}, \qquad \left\{\tilde{S}_m^a, \tilde{S}_n^{b\dagger}\right\} = \delta^{ab}\delta_{mn}, \qquad \left\{S_0^a, S_0^{b\dagger}\right\} = \delta^{ab}. \tag{5.29}$$

The massless fermions $S_{L,R}^A$ have the standard Fourier expansion in terms of the corresponding fermion modes $S_n^A, \tilde{S}_n^A$ satisfying $\{S_n^A, S_m^{B\dagger}\} = \delta^{AB}\delta_{nm}$.

We shall define the vacuum $|0\rangle$ as a state annihilated by $a_n^i, \tilde{a}_n^i$ and $S_n^a, \tilde{S}_n^a$, with $n = 1, \ldots, \infty$, as well as by $a_0^i$ and $S_0^a$. This is a natural definition, since, in particular, it ensures the regularity at $-\infty$ of the euclidean time axis: the physical states created by acting with vertex operators on $|0\rangle$ at $\tau \to i\infty$ must be regular, and therefore the coefficients of $e^{-2iw_n\tau}$ in the Fourier expansion must annihilate the vacuum.[15] The representation of the Clifford algebra satisfied by the fermionic zero modes is constructed by acting by $S_0^\dagger$ on the Clifford vacuum state.

The light-cone hamiltonian can be written as

$$\begin{aligned} H = -p_u = \frac{1}{16\pi\alpha' p_v}\int_0^\pi d\sigma \Big( & \partial_0 x_i \partial_0 x_i + \partial_1 x_i \partial_1 x_i + 4\mathrm{m}^2 x_i^2 + \partial_0 x_\alpha \partial_0 x_\alpha + \partial_1 x_\alpha \partial_1 x_\alpha - \\ & - iS_R^a \partial_1 S_R^a + iS_L^a \partial_1 S_L^a + 4i\mathrm{m} S_L^a \Lambda_{ab} S_R^b - \\ & - iS_R^A \partial_1 S_R^A + iS_L^A \partial_1 S_L^A \Big). \end{aligned} \tag{5.30}$$

By using the equations of motion for the fermions, we can put it in the form

$$\begin{aligned} H = \frac{1}{16\pi\alpha' p_v}\int_0^\pi d\sigma \Big( & \partial_0 x_i \partial_0 x_i + \partial_1 x_i \partial_1 x_i + 4\mathrm{m}^2 x_i^2 + \partial_0 x_\alpha \partial_0 x_\alpha + \partial_1 x_\alpha \partial_1 x_\alpha \\ & + iS_R^a \partial_0 S_R^a + iS_L^a \partial_0 S_L^a + \\ & + iS_R^A \partial_0 S_R^A + iS_L^A \partial_0 S_L^A \Big). \end{aligned} \tag{5.31}$$

This gives

$$H = -p_u = \frac{p_\alpha^2}{4p_v} + \frac{1}{2\alpha' p_v}\mathcal{H}, \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_L + N_R^0 + N_L^0, \tag{5.32}$$

[15] The same definition of the vacuum was assumed in [14]. As discussed in [15], the definition of the vacuum state in the zero-mode sector is not, in fact, unique, with different definitions corresponding to different orderings of the states in the zero-mode supermultiplet.



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where we have defined ($w_n = \sqrt{\mathrm{m}^2 + n^2},\ n = 0, 1, 2, \ldots$)

$$\mathcal{H}_0 = w_0 \left( a_0^{i\dagger} a_0^i + S_0^{a\dagger} S_0^a \right), \tag{5.33}$$

$$\mathcal{H}_R = \sum_{n=1}^{\infty} w_n \left( a_n^{i\dagger} a_n^i + S_n^{a\dagger} S_n^a \right), \qquad \mathcal{H}_L = \sum_{n=1}^{\infty} w_n \left( \tilde{a}_n^{i\dagger} \tilde{a}_n^i + \tilde{S}_n^{a\dagger} \tilde{S}_n^a \right), \tag{5.34}$$

$$N_R = \sum_{n=1}^{\infty} n \left( a_n^{i\dagger} a_n^i + S_n^{a\dagger} S_n^a \right), \qquad N_L = \sum_{n=1}^{\infty} n \left( \tilde{a}_n^{i\dagger} \tilde{a}_n^i + \tilde{S}_n^{a\dagger} \tilde{S}_n^a \right), \tag{5.35}$$

$$N_R^0 = \sum_{n=1}^{\infty} n \left( a_n^{\alpha\dagger} a_n^\alpha + S_n^{A\dagger} S_n^A \right), \qquad N_L^0 = \sum_{n=1}^{\infty} n \left( \tilde{a}_n^{\alpha\dagger} \tilde{a}_n^\alpha + \tilde{S}_n^{A\dagger} \tilde{S}_n^A \right). \tag{5.36}$$

The physical states must satisfy the constraint (expressing invariance under translations in $\sigma$)

$$N_R + N_R^0 = N_L + N_L^0 \,. \tag{5.37}$$

Note that with the definition of the vacuum state we have assumed, the normal ordering constants have cancelled between the fermionic and bosonic contributions, both in the string-oscillator and the zero-mode sectors. The same hamiltonian was given in [14].

The string states are then constructed by acting by the creation operators on the vacuum. Apart from the translational contribution $p_\alpha^2/4p_v$ in 4 directions $x_\alpha$ this results in a discrete spectrum of states.

### 5.3 Equivalence of the supergravity parts of the spectra of the NS-NS and R-R models

The NS-NS and R-R plane-wave models correspond to the two S-dual 3-form backgrounds. The S-duality is an exact global symmetry of the type-IIB supergravity [33], and thus the spectra of fluctuations of the supergravity fields near the two S-dual backgrounds must be in one-to-one correspondence.[16] That implies that the zero-mode (i.e. the $\alpha' \to 0$) parts of the spectra (4.17) and (5.32) of the two string models we have discussed above must be equivalent.

That the two string models are indeed equivalent in the point-particle ($\alpha' \to 0$) limit follows directly from the form of the corresponding light-cone string actions. The bosonic parts of the two actions (cf. (2.11) and (5.1), (5.6)) have the same target space metric part, while the extra $B_{mn}$-coupling term in the bosonic action does not contribute in the point-particle limit. The fermionic actions (2.19) and (5.7) are related by a rotation of the fermions: the $H_{mnk}$ and $F_{mnk}$ terms in the covariant derivative (5.4) are related by the transformation of $\theta^I$ (or $S_L, S_R$) that rotates the matrix $\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ into $\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; the difference between the kinetic terms of the "left" and "right" fermions disappears in the particle limit ($\partial_+ \to \partial_- \to \frac{1}{2}\partial_0$).

It is easy to see then the two zero-mode partition functions $Z_0 = \mathrm{Tr} e^{-\beta\mathcal{H}_0}$ coincide, and are equal to $Z_0 = 16(\frac{1+e^{-\beta \mathrm{m}}}{1-e^{-\beta \mathrm{m}}})^4$. The bosonic contributions are obviously the same. The fermionic contribution follows easily from (5.33). To reproduce it in the NS-NS case one

[16]The same applies of course to the case of the $AdS_3 \times S^3$ NS-NS and R-R backgrounds.

may use the GS formulation where the action is given by (2.19) and the $\mathcal{H}_0$ part of (4.17) is $\mathcal{H}_0 = \mathrm{m}(\hat{J}_L - \hat{J}_R)$ where in GS representation the zero-mode parts of the angular momentum operators $J_{L,R}$ are given by the $\sim S_0(\gamma^{12} + \gamma^{34})S_0$ combinations. Doing the same splitting into $(S^a, S^A)$ as above (cf. (5.15)) and rotating the fermions one arrives at the same result for $Z_0$ as in the R-R case.

The equality between the supergravity parts of the respective partition functions shows that in the supergravity sector of the the NS-NS and R-R models there is an equal number of states for each value of the light-cone energy. It is instructive to see directly from the explicit form of the expressions for the NS-NS (4.17) and the R-R spectra (5.33) how different states are mapped into each other. From (4.17) we get (setting $\hat{N}_L = \hat{N}_R = 0$ and omitting the free translational part)

$$\mathcal{H}_0^{\mathrm{NS-NS}} = \mathrm{m}\left(\hat{J}_L - \hat{J}_R\right). \tag{5.38}$$

Using eq. (4.2), we find that the eigenvalues are

$$\mathcal{E}_0^{\mathrm{NS-NS}} = \mathrm{m}\,(l_{1L} + l_{1R} + l_{2L} + l_{2R}) + k_{\mathrm{NS-NS}}\,, \quad k_{\mathrm{NS-NS}} \equiv \mathrm{m}\,(2 + s_{1L} + s_{2L} - s_{1R} - s_{2R})\,, \tag{5.39}$$

where $l_{L,R} = 0, 1, 2, \ldots$, and (for states with $\hat{N}_L = \hat{N}_R = 0$) $s_{L,R}$ can take values $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. They do not take these values *independently*, because of the condition $\hat{N}_L = \hat{N}_R = 0$. For example, in the NS-NS sector, where $N_L = 1$, $N_R = 1$, their possible values satisfy the condition $|s_{1R} \pm s_{2R}| \leq 1\,, |s_{1L} \pm s_{2L}| \leq 1$. On the other hand, the R-R expression (5.33) is

$$\mathcal{H}_0^{\mathrm{R-R}} = \mathrm{m}\left(a_0^{i\dagger} a_0^i + S_0^{a\dagger} S_0^a\right). \tag{5.40}$$

The occupation number of the modes $a_0^i$, $i = 1, 2, 3, 4$, can be any value from 0 to $\infty$, so we may call them $l_{1L}$, $l_{1R}$, $l_{2L}$, $l_{2R}$. Let us call $s_0^a$ the occupation number of the fermion modes $S_0^a$, where $s_0^a = 0, 1$. Thus the energy eigenvalues are

$$\mathcal{E}_0^{\mathrm{R-R}} = \mathrm{m}\,(l_{1L} + l_{1R} + l_{2L} + l_{2R}) + k_{\mathrm{R-R}}\,, \qquad k_{\mathrm{R-R}} \equiv \mathrm{m}\left(s_0^1 + s_0^2 + s_0^3 + s_0^4\right). \tag{5.41}$$

It is easy to see that the contributions $k_{\mathrm{R-R}}$, $k_{\mathrm{NS-NS}}$ indeed take the same values for the different states of the supergravity part of the spectrum. For $k_{\mathrm{R-R}}$, the possible values are 0, m, 2m, 3m, 4m. For $k_{\mathrm{NS-NS}}$, the possible values are indeed the same. The minimal $k_{\mathrm{NS-NS}}$ is for the states with $s_{1R} = 1$, $s_{2R} = 0$, $s_{1L} = -1$, $s_{2L} = 0$, and other three similar states obtained by exchanging the indices $1 \leftrightarrow 2$. The maximal value of $k_{\mathrm{NS-NS}}$ is found for similar states with reverse sign of the spin. $k_{\mathrm{NS-NS}}$ takes only integer values, because $s_{1R} + s_{2R}$ and $s_{1L} + s_{2L}$ are always integer. Thus one sees that the possible values are $k_{\mathrm{NS-NS}} = 0$, m, 2m, 3m, 4m. To find multiplicities, one should also take into account the extra free oscillators. In the NS case, the multiplicity counting is more complicated, but the above proof that $Z_0^{\mathrm{R-R}} = Z_0^{\mathrm{NS-NS}}$ implies that multiplicities are the same. Thus we conclude that the supergravity parts of the NS-NS and R-R spectra match exactly, as required by S-duality.



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## 6. Asymptotic density of states in plane-wave string models

The spectra of the models discussed above have several interesting features. Here we will discuss the corresponding density of states at given light-cone energy. We shall evaluate its asymptotic behavior at large energy. This may be viewed also as a first step towards a study of thermodynamics of the corresponding string ensembles.[17]

We shall use the dimensionless light-cone energy $\mathcal{E}$ measured in units of $2\alpha' p_v$, i.e. the eigenvalue of $\mathcal{H}$ in (4.7), (4.16), (5.32). In general, the total number $d_{\mathcal{E}}$ of physical string states with given light-cone energy $\mathcal{E}$ can be found from

$$Z = \sum_{\mathcal{E}} d_{\mathcal{E}} \nu^{\mathcal{E}} = \sum_{\mathcal{E}} d_{\mathcal{E}} e^{-\beta\mathcal{E}} \,, \tag{6.1}$$

$$d_{\mathcal{E}} = \frac{1}{2\pi i} \oint \frac{d\nu}{\nu^{\mathcal{E}+1}} Z(\nu) \,, \qquad \nu \equiv e^{-\beta} \,. \tag{6.2}$$

Here $\beta$ is dimensionless "inverse temperature" (related to the standard temperature by $\beta \to \beta/2\alpha' p_v$) and the partition function $Z$ is defined by[18]

$$Z = \mathrm{Tr}\left(e^{-\beta\mathcal{H}}\right) = \int_0^{2\pi} d\lambda \mathrm{Tr}'\left(e^{-\beta\mathcal{H}} e^{i\lambda(\hat{N}_R - \hat{N}_L)}\right) \,. \tag{6.3}$$

The integral over $\lambda$ imposes the level matching constraint (the analog of (5.37)), i.e. the trace Tr is the sum over the physical states, whereas Tr$'$ is the sum over all states of the Fock space.[19]

Since we shall be interested in the $\mathcal{E} \to \infty$ asymptotic form of $d_{\mathcal{E}}$, it will be sufficient to evaluate $Z$ for $\beta \to 0$ (i.e. $\nu \to 1$). Having found $Z$, the integral in (6.2) can be computed in a saddle point approximation.

### 6.1 Models with R-R background

We shall first consider the plane-wave models with the 3-form and 5-form R-R backgrounds. In the case of the 3-form model with the spectrum (5.32) we obtain

$$Z = Z_0 Z_{str} \,, \qquad Z_0 = 16 \left(\frac{1+e^{-\beta \mathrm{m}}}{1-e^{-\beta \mathrm{m}}}\right)^4 \,, \tag{6.4}$$

$$Z_{str} = \int_0^{2\pi} d\lambda \prod_{n=1}^{\infty} \left| \frac{1+e^{-\beta w_n + i\lambda n}}{1-e^{-\beta w_n + i\lambda n}} \right|^8 \prod_{n=1}^{\infty} \left| \frac{1+e^{-\beta n + i\lambda n}}{1-e^{-\beta n + i\lambda n}} \right|^8 \,. \tag{6.5}$$

[17] To set up thermodynamics analysis may be subtle (in particular, the plane-wave background does not admit a straightforward euclidean-time continuation – the plane-wave metric (5.1) becomes complex). Here we shall simply study the number of string states with given light-cone energy at fixed $p^u$.

[18] This $Z$ is the light-cone gauge ("transverse-mode") partition function of a single string. To obtain a thermal partition function for a gas of strings in flat space [34] one needs to integrate over $p^+ \equiv p^u = 2p_v$, i.e. to compute $Z_1(\beta) = \int_0^\infty dp_v e^{-\beta p_v} Z(\frac{\beta}{2\alpha' p_v})$ (related to the trace of $e^{-\beta p_0}$, $p_0 = p_v - p_u = p_v + H$) and then to further "exponentiate" it accounting for the statics of states, $\mathrm{Z}(\beta) = \exp[\frac{1}{2}\sum_{r=1}^\infty \frac{1}{r}(1-(-1)^r) Z_1(r\beta)]$.

[19] In the flat space case ($f = 0$) the constraint can be imposed at the end of the calculation by first treating the left and right sectors separately and then setting $\mathcal{H}_R = N_R$ equal to $\mathcal{H}_L = N_L$ in the definition of $d_{\mathcal{E}_R,\mathcal{E}_L}$. Here we need to impose the level matching constraint by integrating over the Lagrange multiplier $\lambda$ since $\mathcal{H}_{R,L}$ are no longer equal to $N_{R,L}$ (cf. (5.33), (5.35)).

In the flat space limit $f \to 0$, i.e. $w_n \to n$, (6.5) becomes the standard superstring partition function which can be expressed in terms of $\theta$-functions. In computing the zero-mode contribution $Z_0$ we have omitted the translational zero-mode term $p_\alpha^2/4p_v$ in (5.32). The factor 16 in $Z_0$ accounts for the degeneracy in the zero mode part of the "free" sector $(S^A, x^\alpha)$. The zero-mode contribution gives $Z_0(\beta \to 0) \to 16(2/\beta\mathrm{m})^4$.

To evaluate the string-mode contribution $Z_{str}$ in the "large temperature" limit $\beta \to 0$ we shall first ignore the $\lambda$-dependence (i.e. ignore the level matching constraint) and discuss its effect later. Let us formally define (cf. (6.5))

$$Z_R = Z_L = \prod_{n=1}^{\infty} \left( \frac{1+e^{-\beta w_n}}{1-e^{-\beta w_n}} \right)^4 \prod_{n=1}^{\infty} \left( \frac{1+e^{-\beta n}}{1-e^{-\beta n}} \right)^4 , \tag{6.6}$$

and consider the following "bosonic" factor in (6.6): $\prod_{n=1}^{\infty}(1-e^{-\beta w_n}) \equiv e^K$, where

$$K(\beta, \mathrm{m}) \equiv \sum_{n=1}^{\infty} \log \left(1 - e^{-\beta w_n}\right) = - \sum_{n,k=1}^{\infty} \frac{1}{k} e^{-\beta k w_n} . \tag{6.7}$$

To extract the leading $\beta \to 0$ behavior, we have to sum over $n$ and identify the terms which diverge as $\beta \to 0$. Let us expand $w_n = \sqrt{n^2+\mathrm{m}^2} \cong n + \mathrm{m}^2/2n + O(\mathrm{m}^4/n^3)$. Then

$$e^{-\beta k w_n} \cong e^{-\beta k n} e^{-\frac{\beta k \mathrm{m}^2}{2n}} \cong e^{-\beta k n} \left( 1 - \frac{\beta k \mathrm{m}^2}{2n} + \cdots \right) , \tag{6.8}$$

where dots represent terms which become $\beta$-independent in the limit of small $\beta$. Using the symmetry between the sums over $k$ and $n$ we get

$$K \cong - \sum_{n,k=1}^{\infty} e^{-\beta k n} \left( \frac{1}{k} - \frac{\beta \mathrm{m}^2}{2n} + \cdots \right) = - \left( 1 - \frac{1}{2}\beta \mathrm{m}^2 \right) \sum_{k=1}^{\infty} \frac{e^{-\beta k}}{k\,(1-e^{-\beta k})} + \cdots , \tag{6.9}$$

so that for $\beta \to 0$

$$K \cong -\frac{\pi^2}{6\beta} + \frac{\pi^2 \mathrm{m}^2}{12} + O\left(\mathrm{m}^4\right) + O(\beta) . \tag{6.10}$$

We have used that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. Thus the leading term at small $\beta$ is independent of m, while the subleading contribution does not depend on $\beta$.

In general, $K$ in (6.7) can be written as a series of Bessel functions. The resulting expression has simpler form if the zero mode contribution (see (6.4)) is included in the sum. Using the Poisson formula, we get

$$\sum_{n=-\infty}^{\infty} \log \left(1 - e^{-\beta w_n}\right) = \sum_{k=-\infty}^{\infty} \int dx e^{2\pi i x k} \log \left(1 - e^{-\beta w_x}\right) , \qquad w_x = \sqrt{x^2 + \mathrm{m}^2} . \tag{6.11}$$

Expanding the logarithm and performing the integration we obtain

$$\sum_{n=-\infty}^{\infty} \log \left(1 - e^{-\beta w_n}\right) = -2 \sum_{k=-\infty}^{\infty} \sum_{p=1}^{\infty} \frac{\mathrm{m}\beta}{\sqrt{\beta^2 p^2 + 4\pi^2 k^2}} \mathrm{K}_1 \left( \mathrm{m} \sqrt{\beta^2 p^2 + 4\pi^2 k^2} \right) , \tag{6.12}$$

where $\mathrm{K}_1$ is the modified Bessel function.

To find the explicit m dependence of $K$ in the small $\beta$ limit let us note that

$$\frac{\partial K}{\partial \mathrm{m}} = \beta \mathrm{m} \sum_{n=1}^{\infty} \frac{e^{-\beta w_n}}{w_n(1-e^{-\beta w_n})} \underset{\beta\to 0}{\longrightarrow} \mathrm{m} \sum_{n=0}^{\infty} \frac{1}{w_n^2} = \frac{\pi}{2} \coth \pi \mathrm{m} - \frac{1}{2\mathrm{m}} \,, \tag{6.13}$$

so that

$$K(\beta, \mathrm{m}) \cong K_0(\beta) + \frac{1}{2} \log \frac{\sinh \pi \mathrm{m}}{\pi \mathrm{m}} + \cdots \,.$$

Expanding in powers of m, one finds

$$e^K \cong e^{K_0(\beta)} \left( \frac{\sinh \pi \mathrm{m}}{\pi \mathrm{m}} \right)^{1/2} \cong e^{K_0} \left( 1 + \frac{\pi^2 \mathrm{m}^2}{12} + \cdots \right) ,$$

where the second $O(\mathrm{m}^2)$ term corresponds to the correction derived in (6.10). The factor $e^{K_0(\beta)}$ is the standard $\beta$ dependent factor of the flat-space ($\mathrm{m} = 0$) theory, with the leading behavior $e^{K_0} \cong \beta^{-1/2}\, e^{-\pi^2/6\beta}$ (cf. (6.10)).

The analog of $K$ (6.7) corresponding to the fermionic contribution in (6.6) is $K_F(\beta, \mathrm{m}) = \sum_{n=1}^{\infty} \log(1 + e^{-\beta w_n})$. It can be analysed in a similar way as in (6.9), (6.13); we find that in this case there is no m-dependent factor in the $\beta \to 0$ limit. As a result, we find

$$Z_R \cong \mathrm{const.} \beta^6 e^{2\pi^2/\beta} \left( \frac{\pi \mathrm{m}}{\sinh \pi \mathrm{m}} \right)^2 . \tag{6.14}$$

It is easy to see that including the dependence on the Lagrange multiplier $\lambda$ in (6.5) does not change the leading dependence on $\beta$. Now instead of (6.8 we have

$$e^{-\beta k w_n + i\lambda k n} \cong e^{-\beta k n + i\lambda k n} e^{-\frac{\beta k \mathrm{m}^2}{2n}} \cong e^{-\beta k n + i\lambda k n} \left( 1 - \frac{\beta k \mathrm{m}^2}{2n} + \cdots \right) .$$

The second $\mathrm{m}^2$ term in the brackets is irrelevant for the leading $\beta \to 0$ behavior: as above, it gives a finite contribution. Thus the density of states as a function of the energy is expected to have the same leading large $\mathcal{E}$ behavior as in the flat-space theory, $d_{\mathcal{E}} \sim e^{4\pi\sqrt{\mathcal{E}}}$. It would be interesting to determine explicitly the m-dependence of the subleading terms in $d_{\mathcal{E}}$.

The same analysis can be repeated for the string model of [13, 14, 15] corresponding to the plane-wave background [16] with the R-R 5-form field of strength $f$. Here the light-cone hamiltonian can be obtained from (5.32) by replacing 4+4 by 8+8 massive bosons and fermions and discarding the 4+4 massless species (so that now $i = 1, \ldots, 8$; $a = 1, \ldots, 8$ and the are no $N^0_{L,R}$ and $p^2_\alpha$ terms in (5.32), (5.37)). The corresponding string spectrum was described in detail in [15]. Explicitly, one finds

$$H \equiv \left(2\alpha' p_v\right)^{-1} \mathcal{H} \,, \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_L \,, \qquad \mathcal{H}_0 = \mathrm{m} \left( a_0^{i\dagger} a_0^i + S_0^{a\dagger} S_0^a \right) , \tag{6.15}$$

$$\mathcal{H}_R = \sum_{n=1}^{\infty} w_n \left( a_n^{i\dagger} a_n^i + S_n^{a\dagger} S_n^a \right) , \qquad \mathcal{H}_L = \sum_{n=1}^{\infty} w_n \left( \tilde{a}_n^{i\dagger} \tilde{a}_n^i + \tilde{S}_n^{a\dagger} \tilde{S}_n^a \right) , \tag{6.16}$$

$$w_n \equiv \sqrt{n^2 + \mathrm{m}^2} \,, \qquad \mathrm{m} = 2\alpha' p_v f \,. \tag{6.17}$$

The $\mathrm{m} = 0$ limit is the flat string theory limit, while the $\alpha' \to 0$ limit is the supergravity (zero-mode) limit where all string oscillation modes become infinitely heavy and decouple.

This hamiltonian has also another regular limit [15]: $\alpha' \to \infty$ or $\mathrm{m} \to \infty$, i.e. the tensionless string limit, where $w_n \to \mathrm{m}$. In this limit there is the same mass scale m at all string levels including the zero-mode one: at all levels there are states with the same mass as at the lowest (supergravity) level so that one cannot naturally decouple the string states from the supergravity states.

Here we find that $Z$ in (6.3) takes the form (cf. (6.4), (6.5))

$$Z = Z_0 Z_{str}\,, \qquad Z_0 = \left(\frac{1+e^{-\beta\mathrm{m}}}{1-e^{-\beta\mathrm{m}}}\right)^8, \qquad Z_{str} = \int_0^{2\pi} d\lambda \prod_{n=1}^{\infty} \left|\frac{1+e^{-\beta w_n + i\lambda n}}{1-e^{-\beta w_n + i\lambda n}}\right|^{16}. \tag{6.18}$$

In the small $\beta$ limit, we get the following estimate for the "unconstrained" right part of $Z_{str}$ (i.e. the analog of (6.6))

$$Z_R \cong \mathrm{const.}\beta^6 e^{2\pi^2/\beta} \left(\frac{\pi\mathrm{m}}{\sinh \pi\mathrm{m}}\right)^4 . \tag{6.19}$$

The corresponding density of states as a function of the energy should then have again the same leading behavior as in flat space $d_{\mathcal{E}} \sim e^{4\pi\sqrt{\mathcal{E}}}$.

### 6.2 Model with NS-NS 3-form background

Let us now compute the asymptotic density of states in the NS-NS model (2.9) with the spectrum (4.15), (4.16). The corresponding type II superstring partition function (6.3) is given by the product of the equal left $Z_L$ and right $Z_R$ sector contributions with[20]

$$\begin{aligned} Z_R(\beta, \mathrm{m}_1, \mathrm{m}_2) &= \mathrm{Tr}\left[e^{-\beta\left(\hat{N}_R - \mathrm{m}_1 \hat{J}_{1R} - \mathrm{m}_2 \hat{J}_{2R}\right)}\right] \qquad (6.20)\\ &= \prod_{n=1}^{\infty} \frac{[1+2e^{-\beta n}\cosh(\frac{1}{2}\beta(\mathrm{m}_1+\mathrm{m}_2)) + e^{-2\beta n}]^2 [1+2e^{-\beta n}\cosh(\frac{1}{2}\beta(\mathrm{m}_1-\mathrm{m}_2)) + e^{-2\beta n}]^2}{(1-e^{-\beta n})^4 [1-2e^{-\beta n}\cosh(\beta\mathrm{m}_1) + e^{-2\beta n}][1-2e^{-\beta n}\cosh(\beta\mathrm{m}_2) + e^{-2\beta n}]} \end{aligned}$$

We used the GS formalism expressions for the angular momentum operators (4.2), where the spin operators are

$$\begin{aligned} s_{1R} &= \sum_{n=1}^{\infty} \left(b^\dagger_{1n+} b_{1n+} - b^\dagger_{1n-} b_{1n-}\right) - \frac{i}{4} S_0 \gamma^{12} S_0 - \frac{i}{2} \sum_{n=1}^{\infty} S^\dagger_n \gamma^{12} S_n\,,\\ s_{1L} &= \sum_{n=1}^{\infty} \left(\tilde{b}^\dagger_{1n+} \tilde{b}_{1n+} - \tilde{b}^\dagger_{1n-} \tilde{b}_{1n-}\right) - \frac{i}{4} \tilde{S}_0 \gamma^{12} \tilde{S}_0 - \frac{i}{2} \sum_{n=1}^{\infty} \tilde{S}^\dagger_n \gamma^{12} \tilde{S}_n\,. \end{aligned}$$

$s_{2R,L}$ are given by similar expressions with $b_{1n} \to b_{2n}$ and $\gamma^{12} \to \gamma^{34}$. Here $b_n, b^\dagger_n$ are the bosonic mode operators in the planes 12 and 34 satisfying $[b, b^\dagger] = 1$. The fermion mode operators satisfy $\{S^{a\dagger}_n, S^b_n\} = \delta^{ab}$, $\{S^a_0, S^b_0\} = \delta^{ab}$. The expression (6.21) can be written also as

$$Z_R(\beta, \mathrm{m}_1, \mathrm{m}_2) = \frac{e^{\frac{\beta}{4}} \sinh(\frac{1}{2}\beta\mathrm{m}_1) \sinh(\frac{1}{2}\beta\mathrm{m}_2) \theta_2^2(\frac{1}{2}\tau(\mathrm{m}_1+\mathrm{m}_2), \tau) \theta_2^2(\frac{1}{2}\tau(\mathrm{m}_1-\mathrm{m}_2), \tau)}{4\kappa^6(\beta) \cosh(\frac{1}{4}\beta(\mathrm{m}_1+\mathrm{m}_2)) \cosh(\frac{1}{4}\beta(\mathrm{m}_1-\mathrm{m}_2)) \theta_1(\tau\mathrm{m}_1, \tau) \theta_1(\tau\mathrm{m}_2, \tau)}\,, \tag{6.21}$$

[20] In contrast to the R-R model, here we will keep the parameters $f_1$ and $f_2$ (i.e. $\mathrm{m}_1$ and $\mathrm{m}_2$) general. As in section 4 we use hats to indicate the quantum operators.

where $\tau = i\beta/2\pi$, $\kappa \equiv \prod_{n=1}^{\infty}(1-e^{-\beta n})$ and $\theta_{1,2}(z,\tau)$ are the Jacobi theta functions. As in (6.6), here we have omitted the zero mode factor (associated with the Landau quantum numbers in (4.2) and the fermion zero modes). Including the zero-mode contribution modifies (6.21) by an extra factor which cancels the hyperbolic sine and cosine functions in (6.21) . Using the property $\theta_1(z+\tau,\tau) = -e^{-i\pi\tau-2i\pi z}\theta_1(z,\tau)$, we see that the partition function (with zero mode contribution included) is periodic in $\mathrm{m}_1, \mathrm{m}_2$ with period 2, i.e. is invariant under $\mathrm{m}_1 \to \mathrm{m}_1+2n_1, \mathrm{m}_2 \to \mathrm{m}_2+2n_2$, with integer $n_1, n_2$. This is a manifestation of the periodicity of the full string theory under such shifts of the magnetic field parameters discussed at the end of section 4 (see also [18, 19]). In the special case $\mathrm{m}_1 = \mathrm{m}_2$ the period of m is 1.

Closely related computation (with $\mathrm{m}_2 = 0$) in the bosonic case was done in [35], where, however, a magnetic field parameter was used only as a Lagrange multiplier for the angular momentum operator, i.e. the density of states did not depend on it.

By using the modular transformation property of the $\theta$-function, one obtains the following behavior at small $\beta$:

$$Z_R(\beta,\mathrm{m}_1,\mathrm{m}_2) \cong \mathrm{const.}\beta^6 e^{2\pi^2/\beta}\frac{\mathrm{m}_1\mathrm{m}_2}{\sin(\pi\mathrm{m}_1)\sin(\pi\mathrm{m}_2)}\,. \tag{6.22}$$

Interestingly, $Z_R$ (6.22) formally coincides (after setting $\mathrm{m}_1 = \mathrm{m}_2$ in (6.22)) with the partition function (6.14) of the R-R spectrum, provided we make the substitution $\mathrm{m} \to i\mathrm{m}$.

The number of states with given energy $\mathcal{E}$ and given angular momenta $J_{1R}, J_{2R}$ can be computed as follows. Let us make the following replacement in (6.22): $\mathrm{m}_1 \to \mathrm{m}_1 + i\frac{k_1}{\beta}$, $\mathrm{m}_2 \to \mathrm{m}_2 + i\frac{k_2}{\beta}$, and then consider the expansion

$$Z_R\left(\beta,\mathrm{m}_1+i\frac{k_1}{\beta},\mathrm{m}_2+i\frac{k_2}{\beta}\right) = \sum_{N_R,J_{1R},J_{2R}} d_{N_R,J_{1R},J_{2R}}\; e^{-\beta(N_R-\mathrm{m}_1J_{1R}-\mathrm{m}_2J_{2R})}e^{ik_1J_{1R}+ik_2J_{2R}}\,. \tag{6.23}$$

Then

$$\begin{aligned} d_{N_R,J_{1R},J_{2R}} &= \int d\beta e^{\mathcal{E}_R\beta}\int_{-\infty}^{\infty} dk_1dk_2e^{-i(k_1J_{1R}+k_2J_{2R})}Z_R\left(\beta,\mathrm{m}_1+i\frac{k_1}{\beta},\mathrm{m}_2+i\frac{k_2}{\beta}\right), \\ &\mathcal{E}_R \equiv N_R-\mathrm{m}_1J_{1R}-\mathrm{m}_2J_{2R}\,. \end{aligned} \tag{6.24}$$

The integrals over $k_1$, $k_2$ can be computed exactly, by using the identity

$$\int_{-\infty}^{\infty} dke^{-ikJ}\frac{k-i\mathrm{m}\beta}{\sinh\left(\pi(k-i\mathrm{m}\beta)/\beta\right)} = \frac{\beta^2e^{\beta\mathrm{m}J}}{2\cosh^2(\beta J/2)}\,. \tag{6.25}$$

The remaining integral over $\beta$ is evaluated as usual in a saddle point approximation.

Combining the left and right sectors and imposing the level matching constraint we get, following [35],

$$d_{\mathcal{E},J_{1R},J_{1L},J_{2R},J_{2L}} = \left(d_{N_R,J_{1R},J_{2R}}\; d_{N_L,J_{1L},J_{2L}}\right)_{N_R=N_L}\,, \tag{6.26}$$

where $d_{N_R,J_{1R},J_{2R}}$ is given by ($d_{N_L,J_{1L},J_{2L}}$ has a similar form)

$$d_{N_R,J_{1R},J_{2R}} = \mathrm{const.}\frac{h_R^{-19/4}}{\cosh^2\left(\frac{\pi J_{1R}}{\sqrt{2h_R}}\right)\cosh^2\left(\frac{\pi J_{2R}}{\sqrt{2h_R}}\right)}\exp\left[\frac{2\pi\left(N_R+h_R\right)}{\sqrt{2h_R}}\right], \tag{6.27}$$

with

$$h_R \equiv N_R - |J_{1R}| - |J_{2R}| \,. \tag{6.28}$$

Note that the dependence of the density of states (6.26), (6.27) on the magnetic fields $\mathrm{m}_1, \mathrm{m}_2$ is effectively hidden inside $N_R = N_L$, since

$$N_R = N_L = \frac{1}{2}\left[\mathcal{E} + \mathrm{m}_1 \left(J_{1R} - J_{1L}\right) + \mathrm{m}_2 (J_{2R} - J_{2L})\right]. \tag{6.29}$$

Only the sector with $J_{1R} = J_{1L}$, $J_{2R} = J_{2L}$ (or with $J_{1R} - J_{1L} + J_{2R} - J_{2L} = 0$ in the $\mathrm{m}_1 = \mathrm{m}_2$ case) has density of states which is insensitive to the values of the magnetic field parameters $\mathrm{m}_1, \mathrm{m}_2$ . In general, the exponent in (6.27) changes as the magnetic parameters are varied.

The total number of states with given energy can be found by integrating (6.26), (6.27) between the Regge trajectories. The integral seems hard to compute in a closed form, but an estimate is made by noting that for any $|J_{1R}|, |J_{2R}| < N_R^{1-\epsilon}$, the contributions of the angular momenta $J_{1R}, J_{2R}$ can be neglected as compared to $N_R$. This holds true in the relevant integration region, since outside this interval the integrand is exponentially suppressed due to the factor coming from the hyperbolic cosine in (6.27). As a result, we find

$$d_{\mathcal{E}} = \mathrm{const.}\, \mathcal{E}^{-15/4} \exp\left(4\pi\sqrt{\mathcal{E}}\right), \tag{6.30}$$

which is the same behavior as in flat space. In the above derivation we have implicitly assumed that $\mathrm{m}_{1,2} < O(\sqrt{N_R})$. This is certainly the case: as mentioned above, this theory is, in fact, periodic in $\mathrm{m}_{1,2}$ with period 2, so that $\mathrm{m}_{1,2}$ can be restricted to the interval $0 < \mathrm{m}_{1,2} < 2$ (in the interval $2 < \mathrm{m}_{1,2} < 4$, the roles of the zero-mode oscillators and some non-zero mode oscillators are exchanged).

## 7. Plane-wave models with reduced supersymmetry

A simple way to generalize the above plane wave models maintaining their solvability but reducing the number of supersymmetries is by adding twists in spatial 2-planes. This amounts to shifting the polar angles of eq. (3.5) as $\varphi_{1,2} \to \varphi'_{1,2} + b_{1,2}\psi$, where $\psi \equiv \psi + 2\pi R$ is a compact coordinate of a circle of radius $R$ (see e.g. [31, 28]). The corresponding background is locally (but, for generic $b_{1,2}$, not globally) equivalent to (3.5). While in the special case $b_1 = b_2$, this procedure breaks only half of the supersymmetry [28], for generic $b_1, b_2$ all supersymmetries are broken. For the fermions, the effect of the twists is equivalent to adding a locally trivial connection.

We will consider two cases: when $\psi$ is one of the extra 4 flat coordinates $x_\alpha$, and the case when $\psi$ is the same as $y = \frac{1}{2}(u + v)$. In the latter case we will need first to generalize the previous discussion to the case when $y$ direction in (4.1) is periodic.

### 7.1 NS-NS model with $\psi = x_5$

Here the metric in (3.7) becomes ($\alpha = 5, 6, 7, 8$)

$$\begin{aligned} ds^2 = {}& dudv - (f_1^2 r_1^2 + f_2^2 r_2^2) du^2 + \\ & + dr_1^2 + r_1^2 (d\varphi_1 + b_1 dx_5)^2 + dr_2^2 + r_2^2 (d\varphi_2 + b_2 dx_5)^2 + dx_\alpha^2 \,. \end{aligned} \tag{7.1}$$

The solution of the corresponding generalization of the string model (3.7) proceeds along the same lines as in the case of the models in [18, 19, 31]: the twists simply modify the boundary conditions for the world-sheet fields.

Let $m$ and $w$ represent the Kaluza-Klein momentum and the winding number associated with the $x_5$ direction. A shortcut way to obtain the spectrum is by using the observation of [31] that the spectrum of the twisted theory can be obtained from the spectrum of the untwisted one by the formal replacements:

$$\begin{aligned} m &\to m - b_1 R\hat{J}_1 - b_2 R\hat{J}_2\,, \\ \hat{N}_R &\to \hat{N}_R - b_1 wR\hat{J}_{1R} - b_2 wR\hat{J}_{2R}\,, \\ \hat{N}_L &\to \hat{N}_L + b_1 wR\hat{J}_{1L} + b_2 wR\hat{J}_{2L}\,. \end{aligned} \tag{7.2}$$

Then from (4.14) we find

$$\begin{aligned} \alpha'\left(E^2 - p_6^2 - p_7^2 - p_8^2\right) =\ & 2\left(\hat{N}_L + \hat{N}_R\right) + \alpha' R^{-2}\left(m - b_1 R\hat{J}_1 - b_2 R\hat{J}_2\right)^2 + \frac{w^2R^2}{\alpha'} + \\ & + \alpha' p_y^2 - 2\alpha'\left(p_y + E\right)\left[f_1\left(\hat{J}_{1R} - \hat{J}_{1L}\right) + f_2\left(\hat{J}_{2R} - \hat{J}_{2L}\right)\right] - \\ & - 2b_1 wR\left(\hat{J}_{1R} - \hat{J}_{1L}\right) - 2b_2 wR\left(\hat{J}_{2R} - \hat{J}_{2L}\right). \end{aligned} \tag{7.3}$$

This model has no supersymmetry if $b_1 \neq b_2$, and has 8 (counting real supercharges) supersymmetries preserved when $b_1 = b_2$.

### 7.2 NS-NS model with $\psi = y \equiv \frac{1}{2}(u+v)$

Here the generalization of the metric in (3.7) is

$$\begin{aligned} ds^2 =\ & dudv - \left(f_1^2 r_1^2 + f_2^2 r_2^2\right) du^2 + dr_1^2 + r_1^2\left[d\varphi_1 + \frac{1}{2}b_1(du+dv)\right]^2 + \\ & + dr_2^2 + r_2^2\left[d\varphi_2 + \frac{1}{2}b_2(du+dv)\right]^2 + dx_\alpha^2\,, \end{aligned} \tag{7.4}$$

where $y = \frac{1}{2}(u+v)$ is assumed to have period $2\pi R$. As was already stressed above, in the case of compact $y$ direction (and thus periodic $u$ and $v$) the string models (2.4) and (2.6) are no longer equivalent for generic $f_{1,2}R$.[21]

Let us first ignore the twists in the two 2-planes and generalize the results of section 4 to the case of compact $y$. Following [19] (i.e. modifying the light-cone gauge condition by an extra $\sigma$-dependent term) one finds that for compact $y$ the spectrum (4.3) of the NS-NS model (2.4) develops the following dependence on $wR$

$$E^2 - p_\alpha^2 = \frac{2}{\alpha'}\left(\hat{N}_L + \hat{N}_R\right) + \frac{m^2}{R^2} + \frac{w^2R^2}{\alpha'^2} - 4\left(\frac{m}{R} + \frac{wR}{\alpha'} + E\right)\left(f_1\hat{J}_{1R} + f_2\hat{J}_{2R}\right). \tag{7.5}$$

[21] The case $f_{1,2}R = n_{1,2} = 0, 1, 2, \ldots$, is special since in this case the coordinate transformation (3.6) is globally defined and the two models are still equivalent.

The spectrum (4.14) of the model (2.6) is replaced in a similar way by

$$E'^2 - p_\alpha^2 = \frac{2}{\alpha'}\left(\hat N_L + \hat N_R\right) + \frac{m'^2}{R^2} - 2\left(\frac{m'}{R} + E'\right)\left[f_1\left(\hat J_{1R} - \hat J_{1L}\right) + f_2\left(\hat J_{2R} - \hat J_{2L}\right)\right] + \\ + \frac{w^2R^2}{\alpha'^2} - 2\frac{wR}{\alpha'}\left[f_1\left(\hat J_{1R} + \hat J_{1L}\right) + f_2\left(\hat J_{2R} + \hat J_{2L}\right)\right]. \qquad (7.6)$$

Here the integers $m, w$ represent the momentum and winding numbers in the direction $y$. The spectrum (7.5) is the extension of the spectrum of the model studied in [19] to the two-parameter $(f_1, f_2)$ case.

The difference between the two models is due to the fact that the coordinate redefinition in (3.6) now introduces a modification in the boundary conditions. The compact version of the model (2.4), (2.18) with the spectrum (7.5) has 12 supersymmetries if $f_1 = \pm f_2$, and none in the opposite case (see discussion below (2.19)). The compact version of the model model (2.6), (2.19) with the spectrum (7.6) has no supersymmetries for generic $f_1, f_2$ and 8 supersymmetries if $f_1 = \pm f_2$. For special values $f_1R = f_2R = n = 0, 1, 2, \ldots$ there is an enhancement of supersymmetry from 8 to 16 supercharges.

Let us now include the twists $b_1, b_2$ in the two polar angles. Let us first consider the spectrum of the $(b_1, b_2)$ twisted version of the model (2.4), (2.18). Using the prescription (7.2) in the spectrum (7.5), we get

$$\alpha' E^2 - \alpha' p_\alpha^2 = 2\left(\hat N_L + \hat N_R\right) + \frac{\alpha'}{R^2}\left(m - b_1R\hat J_1 - b_2R\hat J_2\right)^2 + \frac{w^2R^2}{\alpha'} - \\ - 4\alpha'\left(\frac{m}{R} - b_1\hat J_1 - b_2\hat J_2 + \frac{wR}{\alpha'} + E\right)\left(f_1\hat J_{1R} + f_2\hat J_{2R}\right) - \\ - 2b_1wR\left(\hat J_{1R} - \hat J_{1L}\right) - 2b_2wR\left(\hat J_{2R} - \hat J_{2L}\right). \qquad (7.7)$$

The spectrum of the twisted version of the model (2.6), (2.19) whose metric is (7.4) can then be found using the observation that the corresponding generalizations of the models (2.9) and (3.5) are formally related by: (i) the shift of the twist parameters: $b_1 \to b_1 + f_1, b_2 \to b_2 + f_2$, and (ii) the shift of the energy in (4.12). Indeed, going from (2.4) to (2.6) is equivalent (see (3.6)) to the twist $\varphi_{1,2} \to \varphi_{1,2} + f_{1,2}(y - \mathrm{t})$. The shift by $f_{1,2}\mathrm{t}$ produces the redefinition of the energy (4.12), while the shift by $f_{1,2}y$ is equivalent to the redefinition of the parameters $b_{1,2} \to b_{1,2} - f_{1,2}$. As a result, we find

$$\alpha'\left(E'^2 - p_\alpha^2\right) = 2\left(\hat N_L + \hat N_R\right) + \frac{\alpha'}{R^2}\left(m' - b_1R\hat J_1 - b_2R\hat J_2\right)^2 + \frac{w^2R^2}{\alpha'} - \\ - 2\alpha'\left(\frac{m'}{R} - b_1\hat J_1 - b_2\hat J_2 + E'\right)\left[f_1\left(\hat J_{1R} - \hat J_{1L}\right) + f_2\left(\hat J_{2R} - \hat J_{2L}\right)\right] - \\ - 2wR\left[(b_1 + f_1)\hat J_{1R} - (b_1 - f_1)\hat J_{1L} + (b_2 + f_2)\hat J_{2R} - (b_2 - f_2)\hat J_{2L}\right]. \quad (7.8)$$

The same result can be obtained also by directly applying the prescription (7.2) to the spectrum (7.6). Note that for $f_1 = f_2 = 0$, the spectrum (7.8) reduces to that of [31, 36, 28].

For general values of the parameters all supersymmetries are broken. In the special case of $b_1 = b_2$ and $f_1 = f_2$ (modulo simultaneous change of sign of $b_2$ and $f_2$) the twisted versions of the model (2.4), (2.18), (7.7) and the model (2.6), (2.19), (7.8) have 6 and 4 supersymmetries respectively.



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### 7.3 R-R model with compact $y \equiv \frac{1}{2}(u+v)$

Finally, let us comment on the generalization of the R-R 3-form model (5.1), (5.2) to the case of the compact $y = \frac{1}{2}(u+v)$ when $u$ and $v$ are periodic with period $2\pi R$. In the case of the plane-wave background supported by $F_5$-field similar to (5.2) the formal solution for the 32 Killing spinors [16] is periodic in $u$ with period $2\pi f^{-1}$. That means that all light-cone gauge supersymmetries will be broken unless $fR = n, n = 1, 2, \ldots$[22] In the present case of the $F_3$-background (5.2) for generic $fR$ we will still have 8 supersymmetries preserved, corresponding to translations in the "massless" $S^A_{L,R}$ fermionic directions (the number of supersymmetries will be enhanced to 16 if $fR$ is integer). This is indeed the same amount of supersymmetry which is present in the "compact" version of the NS-NS model in the corresponding "S-dual" $H_3$-background discussed in section 2 and above.

The "compact" version of the R-R model can be solved by choosing (as in the NS-NS case [18]) the following generalization of the light-cone gauge: $u = 2\alpha' p^u \tau + 2wR\sigma$, where $w = 0, 1, 2, \ldots$ is the string winding number and $p^u = m/R + E$, $m = 0, 1, 2, \ldots$ (cf. (4.5)). The fermionic action in the gauge $\Gamma^u \theta^I = 0$ is still quadratic and given by (5.3) with $\partial_a x^m \to \partial_a u$. Assuming $\partial_\pm u \neq 0$ and rescaling the fermions to absorb $\partial_+ u$ and $\partial_- u$ factors, the bosonic and fermionic parts of the action then take the same form as in (5.6), (5.7), but now with $\mathrm{m}^2 = f^2 \partial_+ u \partial_- u = f^2[(\alpha' p^u)^2 - w^2 R^2]$. The spectrum is then again given by (5.32), now with $4p_u p_v + p_\alpha^2 \to -E^2 + (\frac{m}{R})^2 + \frac{(wR)^2}{\alpha'} + p_\alpha^2$ (cf. (7.8)) and with the constraint (5.37) being $N_R + N_R^0 - N_L - N_L^0 = mw$.

## 8. Conclusions

Let us summarize the main results of this paper. We investigated the plane-wave limits of the NS-NS and R-R $AdS_3 \times S^3$ backgrounds and have shown that the resulting string models are explicitly solvable.

In the NS-NS case, the plane-wave model is equivalent to a direct generalization of the NW model. Its solution can be obtained as a special case of a solution of the "magnetic" model in [18, 19]. The spectrum is supersymmetric and, in addition to the usual level number operators $\hat{N}_R$, $\hat{N}_L$, it involves the angular momentum operators $\hat{J}_{1R,1L}$, $\hat{J}_{2R,2L}$ associated with rotations in the two planes 12 and 34 (i.e. shifts of the angles $\varphi_{1,2}$ in the original $AdS_3 \times S^3$ geometry (3.1)). In the compact $u, v$ case, this model may be interpreted as describing a constant magnetic field background in closed string theory.

In the R-R case, the string model can be solved in the light-cone gauge in terms of free massive and massless world-sheet bosons and fermions. We have worked out the operator quantization in detail, determining the hamiltonian and physical states in terms of the creation/annihilation operators corresponding to the Fourier modes of the fields.

We have studied, both in the R-R and NS-NS cases, the asymptotic density of states, finding similar leading large light-cone energy behavior as in flat space. In the NS-NS

[22] Without imposing the light-cone gauge one still finds 16 Killing spinors by restricting the constant spinor by the condition $(\Gamma_{1234} + \Gamma_{5678})\epsilon_0 = 0$. This condition has no solutions in the light-cone gauge, as follows from $\Gamma^v \epsilon_0 = 0$ together with MW property. However, it is not clear how to realize this supersymmetry at the level of string theory of [13].

model, we obtained an explicit formula for the density of states with given energy and angular momentum components. It can also be interpreted as a density of states of a string in magnetic field.

We also discussed solvable plane-wave modes with reduced or completely broken supersymmetry. The corresponding plane wave backgrounds were constructed by shifting the polar angles, $\varphi_s \to \varphi_s + b_s\psi$, in the same way as in the Melvin model [31]. This produces a "twisted" identification of space-time points under $\psi \to \psi + 2\pi R$. The resulting models include examples with $0, 4, 6, 8$ or $12$ real supersymmetries. Their spectra contain as particular cases the spectrum of the uniform magnetic field model of [19], as well as the spectrum of the magnetic flux tube model of [31] and its supersymmetric generalizations studied in [36, 28].

We have seen that the NS-NS and R-R plane-wave string models corresponding to the two S-dual 3-form backgrounds have equivalent supergravity parts of their spectra. In general, the S-duality involves transforming the supergravity background, inverting string coupling and interchanging fundamental strings with D-strings, and thus it does not a priori imply any relation between the stringy parts of the two weakly-coupled fundamental string spectra.

There are a number of open problems and further directions. One may also determine massless vertex operators, compute simplest correlation functions and compare them with the corresponding expressions in $AdS_3 \times S^3$ theory before the plane-wave limit (cf. [37]). Using the light-cone gauge, one may also solve explicitly for the open-string spectrum, determining possible D-brane configurations and comparing with previous results, e.g., for D-branes in the NW model [38].

As is well known (see, e.g., [39]), the $AdS_3 \times S^3$ background is the near-horizon region of the NS5+NS1 or D5+D1 system. Adding momentum (wave) along the string direction leads to an extremal 5-d black hole with regular horizon and finite entropy, which can be reproduced by counting D-brane BPS states at weak coupling and using a non-renormalization property of the entropy. It would be interesting to investigate which part of this picture survives taking the plane-wave limit, and, in particular, whether the knowledge of the full string spectra for the plane-wave models can be useful for the study of some aspects of the black-hole physics, thus complementing implicit (supersymmetry-based) D-brane methods.

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